CARAVAN'S

# MATHEMATICAL TECHNIQUES

For
B.A. / B.Sc.

by
Dr. Karamat H. Dar
Irfan-ul-Haq M. Ashraf Jajja

Caravan Book House-Lahore

Muhammad Saad

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*by*

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## PREFACE TO 3RD EDITION

Al- "Hamdo-lillah", the 3rd edition is out for the readers. The delay in this edition is apologized, but the authors feel happy in presenting this edition with the following mentionable features:

(a) The content has been improved by introducing the following topics:

*(i)* Orthogonal trajectories in chapter 7.

*(ii)* Solution of the differential equations with undetermined coefficients in chapter 8.

*(iii)* The detailed discussion on permutation groups in chapter 2.

(b) The proofs of some of the results have been revised in the light of the suggestions received from our colleagues, friends and the readers.

(c) In order to elaborate some of the concepts more, new solved examples have been added in the content and a few unsolved, problems have been inserted in some of the exercises.

(d) The authors do not claim mistake-free content but they claim of putting hard work to reduce the errors and insert the omissions in order to improve upon it.

(e) Unnecessary matter has been deleted after careful examination and discussion with our colleagues and friends.

We feel it our duty to acknowledge the cooperation of our colleagues, friends and the readers in general who rendered their sincere cooperation and gave their moral support to us. The names of Mr. Mohammad Akram and Prof. Arif Javed in this regard are mentionable in particular.

The authors owe thanks to the Caravan Book House (Publishers) and Mr. Shahbaz Muzammil whose efforts have made it possible to present it to the readers in this form.

At the last and not the least at any standard, we thank, in anticipation to all those who will communicate their suggestions to the authors / publishers, for the improvement of this book.

Dated at Lahore the 3rd Sept., 1998

Authors

## Preface to the Second Edition

Although, the plan and most of the features of the original manuscript have been retained in this edition but some addition's deletions and refinements have been made to increase clarity and achieve satisfaction to the liking of the readers. Notable changes are made through reorganization of exercise 4.1 and insertion of some additional problems in exercise 4.5. A few solved examples have been added wherever thought necessary.

We are indebted and deeply grateful to all our colleagues and well wishers who contributed to the improvement of the present edition either by writing us their words of encouragement, by constructive criticism or by helpful remarks and suggestions. In this regard we feel thankful, in particular to Sh. Mohammad Hafeez, Dr. Shamim Arif, M. Munir Abdi, Nazir Ahmad Cheema, Abdul Hameed, Mohammad Alam, Mohammad Saeed, Mohammad Mudassir, Abdul Karim, Mohammad Mushtaq, Dr. Sabir H. Shah, Abdul Rauf, Abdul Wahab, Obaid Ullah, Miss Shama Javed, Miss Abida, Mohammad Rafaq Siddqui, Abdur Raheem, Mohammad Nawaz Shina and Mohammad Imtiaz.

Lahore:
Sept., 1, 1992

Authors

## Preface to the First Edition

Our long experience of teaching mathematics developed us with the feeling of necessity of bringing some competition in the quantity and quality of good text books at the degree level. Probably no book is ever written without a lot of help and encouragement. In putting forward a humble attempt in the shape of this book, the motivation and help is provided by our sincere colleagues and teachers of the degree classes. This book comprises the introduction of some of the essential mathematical techniques for an easy understanding of the various topics. It provides the reader a foundation for further study in a broad variety of mathematical disciplines. The content being restricted to the current syllabus recommended for the B.A./B.Sc. degree classes of various local universities, has been presented in a simple manner without ignoring the mathematical vigour of the concepts. The subject matter is supported by liberal typical worked out examples followed by well-arranged exercises selected from standard books. In fact, the students are introduced, in this book, to the process of abstraction as a way of making it easier to conceptualize and cope with the problems. Every care has been taken to avoid printing and other mistakes but perfection cannot be claimed. So any intimation of errors and suggestions for improvement will be duly appreciated and acknowledged.

The authors owe an immense gratitude to their worthy teachers and respectable authors of the books from which they sought help during the process of learning mathematics at any stage.

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Authors

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$Z_3 = \{0, 1, 2\}$

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# 1

# COMPLEX NUMBER

## 1.1 INTRODUCTION

We assume that the reader is already familiar with the concept of complex number while considering the solution of $x^2 + 1 = 0$. There is no real number satisfying this equation. We then introduce the imaginary number $i = \sqrt{-1}$ by solving this equation getting i and $-i$ as its roots. In order to solve $ax^2 + bx + c = 0$ and in general, the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0,$$

we introduce the complex number $x + iy$ which is the sum of a real number x and an imaginary number iy.

This introduction of complex numbers is not satisfactory, for it makes them appear as objects not existing in reality, that is, imaginary in the literal sense of the word.

So we introduce the complex number by the following axiomatic approach:

### 1.1.1 THE COMPLEX NUMBER SYSTEM

The complex number system is the set $\mathbb{C} = R \times R = \{(a,b) : a, b \in R\}$ whose elements satisfy the following properties;

(i) $(a, b) = (l, m) \Leftrightarrow a = l$ and $b = m$

(ii) $(a, b) + (l, m) = (a + l, b + m)$

(iii) If r is any real number

$$r(a, b) = (ra, rb)$$

(iv) $(a, b) \cdot (l, m) = (al - bm, am + bl)$

The elements of R x R are complex numbers. In a complex number $z = (a, b) \in \mathbb{C}$, $a$ is called the real part of z and b its imaginary part.

In symbols,

$$a = \text{Re}(z), \qquad b = \text{Im}(z)$$

## 1.1.2 ALGEBRAIC ASPECTS

It can be easily verified that the above mentioned operations of addition and multiplication obey all the rules of algebra. i.e. The commutative, associative and distributive laws. The reader has already verified them in intermediate classes. For revision purposes we enlist them below;

### 1. COMMUTATIVE LAWS

$\forall\ z_1, z_2 \in \mathbb{C}$

(i) $z_1 + z_2 = z_2 + z_1$

(ii) $z_1 \cdot z_2 = z_2 \cdot z_1$

### 2. ASSOCIATIVE LAWS

$\forall\ z_1, z_2, z_3 \in \mathbb{C}$

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2)z_3 = z_1(z_2 z_3)$

### 3. DISTRIBUTIVE LAW

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

*Note:* unlike real numbers, no two complex numbers are comparable by the ordering $<$ or $>$. So the statment, that a complex number $z_1 > z_2$ or $z_1 < z_2$ is meaningless for any pair $z_1$, $z_2$ of complex numbers.

The complex numbers (0, 0) and (1, 0) assume the role of additive and multiplicative identities respectively. The inverse operations of addition and multiplications — namely, subtraction and division — can always be performed within $\mathbb{C}$, except for division by (0, 0). This is evidenced by the following formulas;

If $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ and $-z_2 = (-a_2, -b_2)$, then

*(i)* $z_1 - z_2 = z_1 + (-z_2)$

$= (a_1, b_1) + (-a_2, -b_2)$

$= (a_1 - a_2, b_1 - b_2)$

(ii) $\frac{z_1}{z_2} = \left( \frac{a_1a_2 + b_1b_2}{a^2_2 + b^2_2}, \frac{a_2b_1 - a_1b_2}{a^2_2 + b^2_2} \right), \quad z_2 \neq 0$

which can be easily verified.

This can also be verified that $\mathbb{C}$ is closed with respect to addition, subtraction, multiplication and division, except for division by (0, 0); In other words $\mathbb{C}$ forms the structure of a field

$\mathbb{C}$ contains a subset $\mathbb{C}_o = \{(a, 0)/a \in R\}$

We write $(a, 0) \pm (b, 0) = (a \pm b, 0)$

$(a, 0)\,(b, 0) = (ab, 0)$

$\frac{(a, 0)}{(b, 0)} = \left( \frac{a}{b}, 0 \right), \quad b \neq 0$

so by $(a, 0) \longleftrightarrow a \qquad \forall\, a \in R$

the set $\mathbb{C}_0$ is in one-to-one correspondence with the real set R.

Moreover, $(a, b) = (a, 0) + (b, 0)\,(0, 1)$

but $(0, 1)\,(0, 1) = (-1, 0) \longleftrightarrow -1 = i^2$

writing $(a, 0) = a$

$(b, 0) = b$

$(0, 1) = i$

$(a, b) = a + ib$ or $a + bi$

## 1.1.3 GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

The one-to-one (1 – 1) correspondence between $\mathbb{C}$ and $R \times R = R^2$ (say) suggests a geometric interpretation of the complex numbers. All one has to do is to let a point P(x, y) in a rectangular co-ordinate system represent the complex number $z = x + iy$.

In that way every complex number is mapped uniquely onto a specified point of the xy – plane and, vice versa, every point of that plane corresponds to one and only one complex number. The plane involved is called the complex, or Gaussian plane and is denoted by $\mathbb{C}$ or z-plane.

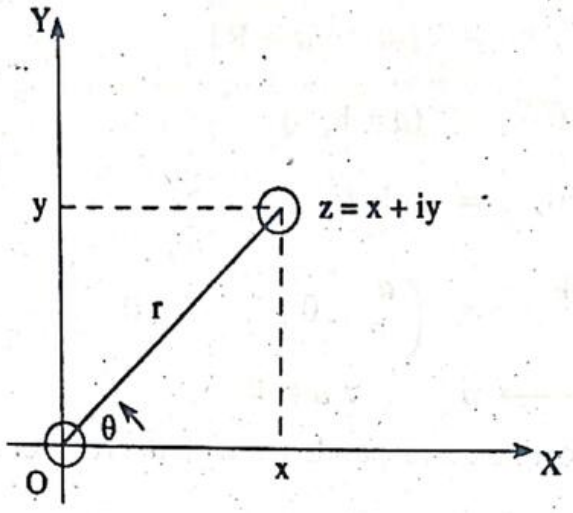
In the $\mathbb{C}$-plane, the x-axis of the co-ordinate system is called the real axis, whereas the y-axis is called the imaginary axis.

## 1.1.4. POLAR REPRESENTATION OF COMPLEX NUMBERS

It is convenient to have representation of z in the polar co-ordinates by introducing (r, θ) as the pair of polar co-ordinates of the point (x, y) in the z-plane.

Thus, $x = r\cos\theta$, and $y = r\sin\theta$

from which we obtain $r = \sqrt{x^2 + y^2} > 0$, $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$




The positive quantity r is called the *modulus* or *absolute value of* z and is denoted by $r = |z|$; while θ is called the *argument* or *amplitude of* z and is denoted by

$$\theta = \arg(z) = \arg(x + iy) = \tan^{-1}\left(\frac{y}{x}\right)$$

Argument of z is not single-valued function, since to any $z \neq 0$ there correspond infinitely many values of arg (z). We shall say that arg (z) is a multi-valued function. i.e. If θ is one of the values of arg (z), we have

$$\arg(z) = \theta + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots.$$

If, $-\pi < \theta \leq \pi$ then θ is called the principal argument of z and is denoted by Arg (z). Therefore, if θ is *principal argument* then we write

$$\theta = \text{Arg}(z)$$

So, from the above figure,

$$z = x + iy = r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

If we use the abbreviation

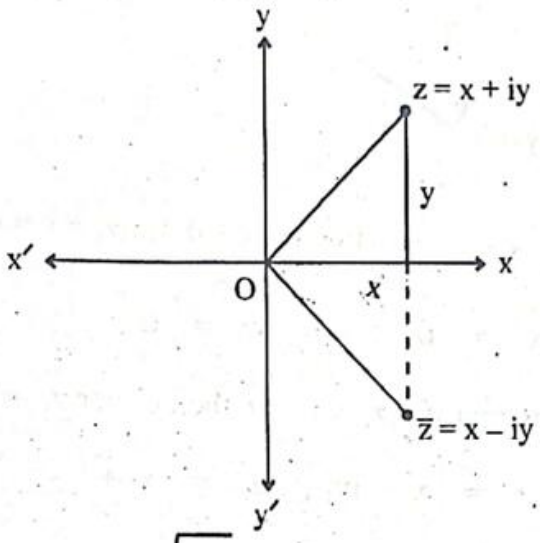
$$\cos\theta + i\sin\theta = \text{cis}\,\theta$$

$$z = r\,\text{cis}\,\theta \Rightarrow |z| = r, \quad |\text{cis}\,\theta| = 1$$

The complex number $x - iy$ is called the *conjugate* of $z = x + iy$.

It will be denoted by $\bar{z}$; i.e., $\bar{z} = x - iy$.

$\bar{z}$ is reflection (or conjugation) of z about the real axis.




Obviously. $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = 1$

Similarly,

$$x = \text{Re}(z) = \frac{1}{2}(z + \bar{z})$$

and, $$y = \text{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

So any Cartesian equation $f(x, y) = 0$ can be written in complex variables. e.g.,

$x^2 + y^2 = a^2$ is changed as $\left(\frac{z + \bar{z}}{2}\right)^2 + \left(\frac{z - \bar{z}}{2i}\right)^2 = a^2$

which after a little simplification takes the shape $z\bar{z} = a^2$.

## PROPERTIES OF COMPLEX CONJUGATION

Let $z_1, z_2 \in \mathbb{C}$ then we can easily demonstrate

*(i)* $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

*(ii)* $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

*(iii)* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad |z_2| > 0$

*(iv)* $\overline{(\bar{z}_1)} = z_1$

*(v)* $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

*(vi)* $|\bar{z}| = |z| = \sqrt{x^2 + y^2}$

*(vii)* $z = 0$ if and only if $|z| = 0$, i.e., $x = 0$ and $y = 0$.

(viii) $z\,\bar{z} = |z|^2$

(ix) $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ if $|z| \neq 0$

**EXAMPLE**

Show that $z=0 \Rightarrow x=0, y=0$

**SOLUTION** $z = 0 \Rightarrow x + iy = 0$. $\text{Re}(z) = x = 0$, $\text{Im}(z) = y = 0$

$\therefore \quad x = 0$ and $y = 0$

**EXAMPLE 2** Show that if $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$

**SOLUTION** Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = 0 = 0 + i0$$

By equating real and imaginary parts

$$x_1x_2 - y_1y_2 = 0$$

and $x_1y_2 + x_2y_1 = 0$

By multiplying first equation by $y_2$ and second equation by $x_2$ and then subtracting, we get

$$-y_1(x_2^2 + y_2^2) = 0 \Rightarrow \text{x's and y's are real}$$

$\therefore \quad y_1 = 0$ or $x_2^2 + y_2^2 = 0 \Rightarrow x_2 = y_2 = 0$

Similarly we can prove that $z_1 = 0$

**EXAMPLE 3** Express the following in polar form $r(\cos\theta + i\sin\theta)$.

(i) $5 + \sqrt{5} + \sqrt{-5}$ (ii) $1 + i\tan a$ (iii) $1 + \sin\alpha + i\cos\alpha$

**SOLUTION** Let $5 + \sqrt{5} + i\sqrt{5} = r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts

$$5 + \sqrt{5} = r\cos\theta \Rightarrow \left(5 + \sqrt{5}\right)^2 = r^2\cos^2\theta$$

and $\sqrt{5} = r \sin\theta \Rightarrow 5 = r^2 \sin^2\theta$

$\therefore\ 25 + 5 + 10\sqrt{5} + 5 = r^2 = 35 + 10\sqrt{5}$

$r = \sqrt{35 + 10\sqrt{5}}$

$\tan\theta = \dfrac{\sqrt{5}}{5 + \sqrt{5}}$, where $\theta$ lies in first quadrant.

$\therefore\ 5 + \sqrt{5} + \sqrt{-5} = \sqrt{35 + 10\sqrt{5}} \left\{ \text{cis}\left(\tan^{-1}\dfrac{\sqrt{5}}{5 + \sqrt{5}}\right)\right\}$

(ii) Let $1 + i\tan a = \dfrac{\cos a + i \sin a}{\cos a} = \dfrac{1}{\cos a}\ [\cos a + i \sin a]$

$\therefore\ r = \sec a$

$\theta = a$

(iii) Let $1 + \sin\alpha + i\cos\alpha = r\ \text{cis}(\theta)$

$\therefore\ 1 + \sin\alpha = r\cos\theta$

$\cos\alpha = r\sin\theta$

$\therefore\ r^2 = 1 + \sin^2\alpha + \cos^2\alpha + 2\sin\alpha$

$= 2(1 + \sin\alpha)$

$= 2(\cos^2 \alpha/2 + \sin^2 \alpha/2 + 2\sin \alpha/2 \cos \alpha/2)$

$\therefore\ r = \sqrt{2}\ [\cos \alpha/2 + \sin \alpha/2]$

$$\tan\theta = \frac{\cos\alpha}{1 + \sin\alpha} = \frac{\cos^2 \alpha/2 - \sin^2 \alpha/2}{(\cos \alpha/2 + \sin \alpha/2)^2} = \frac{1 - \tan \alpha/2}{1 + \tan \alpha/2}$$

$$= \tan(\pi/4 - \alpha/2)$$

$\theta = \pi/4 - \alpha/2$

$\therefore\ 1 + \sin\alpha + i\cos\alpha = r\,\text{cis}(\theta)$

$= \sqrt{2}\,[\cos \alpha/2 + \sin \alpha/2]\ \text{cis}(\pi/4 - \alpha/2)$

$$= 2[\cos \alpha/2 \sin \pi/4 + \sin \alpha/2 \cos \pi/4] \text{ cis } (\pi/4 - \alpha/2)$$

$$= 2\sin (\alpha/2 + \pi/4) \text{ cis } (\pi/4 - \alpha/2)$$

**EXAMPLE 4** Prove

(i) $|z_1 z_2| = |z_1| \cdot |z_2|$

(ii) $|\text{Re}(z)| \leq |z|$

(iii) $|\text{Im}(z)| \leq |z|$

**PROOF** (i) We know $|z| = |\bar{z}|$ and $z\bar{z} = |z|^2$

$$\therefore \quad |z_1 z_2|^2 = (z_1 z_2) \cdot (\overline{z_1 z_2})$$

$$= (z_1 \cdot z_2)(\bar{z}_1 \cdot \bar{z}_2) = z_1 \cdot (z_2 \bar{z}_1) \cdot \bar{z}_2$$

$$= z_1(\bar{z}_1 \cdot z_2) \cdot \bar{z}_2 = (z_1 \cdot \bar{z}_1)(z_2 \cdot \bar{z}_2)$$

$$= |z_1|^2 |z_2|^2$$

$\therefore \quad |z_1 z_2| = |z_1| \cdot |z_2|$ because absolute value of complex number is non-negative.

(ii) $|\text{Re}(z)| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|$

(iii) $|\text{Im}(z)|^2 = y^2$ and $y^2 \leq x^2 + y^2$

$\therefore \quad x^2 + y^2 = |z|^2 \geq |\text{Im}(z)|^2$

or $|\text{Im}(z)| \leq |z|$

We can also prove $|z_1 \cdot z_2 \cdot z_3 \cdots z_n| = |z_1||z_2| \cdot |z_3| \cdots |z_n|$

or briefly $\left| \prod_{k=1}^{n} z_k \right| = \prod_{k=1}^{n} |z_k|$

(The symbol $\prod$ stands for product)

**EXAMPLE 5** Prove $|z_1 + z_2| \leq |z_1| + |z_2|$

**PROOF**

$$|z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$$

because $z_1\bar{z}_2 + z_2\bar{z}_1 = 2(x_1x_2 + y_1y_2)$

$$= 2\mathrm{Re}(z_1\bar{z}_2)$$

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2|, \quad |\bar{z}_2|\,\bar{z}_2 = |z_2|$$

$$= (|z_1| + |z_2|)^2$$

$$\therefore \quad |z_1 + z_2| \leq |z_1| + |z_2|$$

**EXAMPLE 6** $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

**PROOF**

$$|z_1 - z_2|^2 = (z_1 - z_2)\,\overline{(z_1 - z_2)} = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1| \cdot |z_2| = (|z_1| - |z_2|)^2$$

$\therefore \quad |z_1 - z_2| \geq \big||z_1| - |z_2|\big|$, because modulus is non-negative.

$\therefore \quad |\mathrm{Re}(z)|^2 \leq |z|^2 \Rightarrow |\mathrm{Re}(z)| \leq |z|$

### 1.1.5 NOTE

(i) Remember if we are given a complex number $x + iy$ then its principal argument is calculated as follows:

$$\text{Arg}(z) = \text{Arg}(x + iy)$$

$$= \begin{cases} \tan^{-1}\dfrac{y}{x}, & \text{when } x > 0 \\ \pi + \tan^{-1}\dfrac{y}{x}, & \text{when } x < 0, \quad y \geq 0 \\ -\pi + \tan^{-1}\dfrac{y}{x}, & \text{when } x < 0, \; y < 0 \\ \dfrac{\pi}{2}, & \text{when } x = 0, \; y > 0 \\ -\dfrac{\pi}{2}, & \text{when } x = 0, \quad y < 0 \end{cases}$$

$\therefore$ $\arg(z) - \text{Arg}(z) = 2k\pi$ where k is any integer.

(ii) The complex plane or z-plane is usually called Argand diagram.

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore$ $|z_1z_2| = r_1r_2 = |z_1|\,|z_2|$

and $\text{Amp}(z_1z_2) = \theta_1 + \theta_2 = \text{Amp}\,z_1 + \text{Amp}\,z_2$

Also $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \dfrac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$

$= \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$\therefore$ $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$ and

$\text{Amp}\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \text{Amp}\,z_1 - \text{Amp}\,z_2$

**CAUTION** Amplitudes or Arguments calculated by these formulas may or may not be principal Arguments.

**EXAMPLE 7** If $z = x + iy = -1 - i$, then find Arg (z).

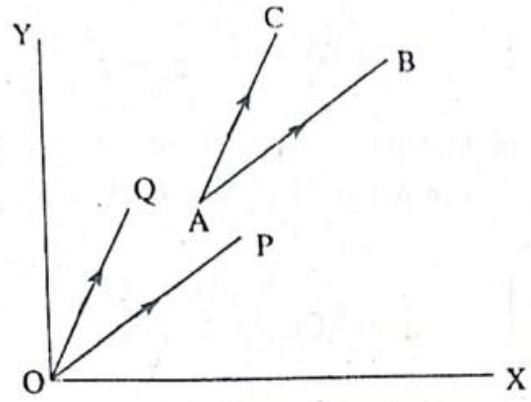
**PROOF** Here $x < 0$, $y < 0$,

therefore $\text{Arg}(z) = -\pi + \tan^{-1}\left(\dfrac{y}{x}\right)$

$= -\pi + \tan^{-1}(1) = -\pi + \pi/4 = -\dfrac{3\pi}{4}$

**EXAMPLE 8** Discuss the Amplitude of $\dfrac{z_3 - z_1}{z_2 - z_1}$

**PROOF** Consider the figure,




Let A, B and C be three points representing complex numbers $z_1$, $z_2$ and $z_3$ respectively.

Draw OP, OQ parallel and equal to AB, AC respectively, then

$$\vec{AB} = \vec{OP} \quad , \quad \vec{AC} = \vec{OQ}$$

$$\text{Amp}\left(\frac{\vec{AC}}{\vec{AB}}\right) = \text{Amp}\left(\vec{AC}\right) - \text{Amp}\left(\vec{AB}\right)$$

$$= \text{Amp}\left(\vec{OQ}\right) - \text{Amp}\left(\vec{OP}\right)$$

$$= \angle XOQ - \angle XOP$$

$$= \angle POQ = \angle BAC$$

$$\text{Now } \vec{OP} = \vec{AB} = \vec{OB} + \vec{AO} = z_2 - z_1$$

$$\text{Similarly } \vec{OQ} = \vec{AC} = \vec{AO} + \vec{OC} = z_3 - z_1$$

$$\therefore \angle BAC = \text{Amp}\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

Suppose $\angle BAC = \pi/2$, then

$$\text{Amp}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \pi/2$$

$$\text{Hence } \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| (\cos \pi/2 + i \sin \pi/2) = \beta i$$

$$\text{where } \beta = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|$$

Thus the line AC is at right angle to AB, if $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.

Again, if A, B, C, D be four points representing complex numbers $z_1$, $z_2$, $z_3$, $z_4$ respectively, and the angle between AB and CD is $\alpha$, then

$$\alpha = \text{amp}\left(\frac{\overrightarrow{CD}}{\overrightarrow{AB}}\right) = \text{amp}\left(\overrightarrow{CD}\right) - \text{amp}\left(\overrightarrow{AB}\right)$$

If $\alpha = 0$ or $\pi$ then AB || CD or collinear

$$\therefore \frac{\overrightarrow{CD}}{\overrightarrow{AB}} = \frac{z_4 - z_3}{z_2 - z_1} = r(\cos\alpha \pm i \sin\alpha) = \pm r$$

Hence if AB ∥ CD, the ratio $\frac{z_4 - z_3}{z_2 - z_1}$ is real.

Remember here complex numbers are treated as free complex vectors.

**EXAMPLE 9** Prove $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

**PROOF** We know $|z_1|^2 = z_1 \bar{z}_1$ and $|z_1| = |\bar{z}_1|$

$$\therefore |z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2 \quad \ldots\ldots \text{(i)}$$

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 + z_2)}$$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 + |z_2|^2 \quad \ldots\ldots \text{(ii)}$$

Adding (i) and (ii)

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$$

**EXAMPLE 10.** Find the locus of z where $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{4}$.

**SOLUTION:**

If $z = x + iy$, then

$$\text{Arg}\left(\frac{z+1}{z-1}\right) = \text{Azg}\,(z+1) - \text{Arg}\,(z-1)$$

$$= \text{Arg}\,(x+1+iy) - \text{Arg}\,(x-1+iy)$$

$$= \text{Tan}^{-1}\left(\frac{y}{x+1}\right) - \text{Tan}^{-1}\left(\frac{y}{x-1}\right)$$

$$\frac{\pi}{4} = \text{Tan}^{-1}\left(\frac{\frac{y}{x+1} - \frac{y}{x-1}}{1 + \frac{y^2}{x^2-1}}\right), \text{ in particular,}$$

$$\text{or } \tan\frac{\pi}{4} = \frac{(yx - y - yx - y)}{x^2-1} \times \frac{x^2-1}{(x^2+y^2-1)}$$

$$1 = \frac{-2y}{x^2+y^2-1}$$

$$x^2 + y^2 - 1 + 2y = 0$$

$$x^2 + (y+1)^2 - 2 = 0$$

or $$x^2 + (y+1)^2 = 2$$

The locus described is a circle with centre ( 0, -1) and radius $\sqrt{2}$

## EXERCISE 1.1

Express the following in the form $x + iy$ :

**1.** (i) $(1+i)^2$ (ii) $(1-i)^2$ (iii) $\left(\frac{1+i}{1-i}\right)^2$ (iv) $\left(\frac{a+ib}{a-ib}\right)^2$

(v) $\frac{2-i}{(1-2i)^2}$

**2.** (i) $\frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\alpha - i\sin\alpha)}$

(ii) $\frac{1+\cos 2\alpha - i\sin 2\alpha}{\cos 2\alpha + i\sin 2\alpha}$ (iii) $\frac{1-\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}$

**3.** Find the modulus and Argument or Amplitude of

(i) $\frac{5+i\sqrt{3}}{4-2i\sqrt{3}}$ (ii) $(1+i)\frac{(2+i)}{(3-i)}$ (iii) $\frac{1+i}{1-i}$ (iv) $1 - \cos\theta + i\sin\theta$

4. If $c^2 + s^2 = 1$, i.e. $c = \cos\theta, s = \sin\theta$ show that

(i) $\dfrac{1 + c + is}{1 + c - is} = c + is$ (ii) $\dfrac{1 + s + ic}{1 + s - ic} = s + ic$

5. Find the modulus and Argument of the sum and the product of the roots of the equations

(i) $z^2 + z + 1 = 0$

(ii) $(2 + 3i)\, z^2 - (5 + i)\, z + (7 + 4i) = 0$

6. If $x + iy = \dfrac{p}{q + \cos\theta + i \sin\theta}$, find the locus of (x, y) when $\theta$ is the parameter, p and q being real numbers.

7. Four points in the complex plane are represented by the complex numbers $z_1, z_2, z_3$ and $z_4$, in order, such that, $z_1 + z_3 = z_2 + z_4$, show that these four points are corners of a parallelogram.

8. Show that the complex numbers $5 + 2i, 7 + 5i, 4 + 7i, 2 + 4i$ are the vertices of a parallelogram. Find the angles and the lengths of its sides.

9. The two opposite vertices of a square are represented by $9 + 12i$ and $-5 + 10i$. Find the other two.

10. Show that the triangle formed in the Argand diagram by the complex numbers $2 + 3i$, $8 + 11i$, $17i$ is right-angled isosceles triangle.

11. Consider Q. 10 above and find the number representing the fourth point which with three given points would form a square.

12. Show that the points $-1, +1, i\sqrt{3}$ are the vertices of an equilateral triangle in the complex plane.

13. Prove that if $z = x + iy$, and $|z+1| = |z-1|\sqrt{2}$, then $x^2 + y^2 - 6x + 1 = 0$.

14. Prove that if $z = x + iy$, and $|2z - 1| = |z - 2|$, then $x^2 + y^2 = 1$.

15. Prove that $|a + \sqrt{(a^2 - b^2)}| + |a - \sqrt{a^2 - b^2}| = |a+b| + |a-b|$

16. Prove that the transformation $z = \dfrac{Z - 1}{Z + 1}$,

changes the circle $x^2 + y^2 = a^2$ into the circle

$(X - 1)^2 + Y^2 = a^2[(X + 1)^2 + Y^2]$

where $z = x + iy$ and $Z = X + iY$.

17. Let $(r, \theta)$ denote the complex number whose modulus is r and argument $\theta$. i.e. $r(\cos\theta + i\sin\theta)$. Then if

$a \equiv (1, \alpha) = \cos\alpha + i\sin\alpha$, $b \equiv (1, \beta) = \cos\beta + i\sin\beta$

$c \equiv (1, \gamma) = \cos\gamma + i\sin\gamma$, and if $a + b + c = 0$,

prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

18. If $|z_1 - z_2| = |z_2 + z_1|$, prove that the difference of the arguments of $z_1$ and $z_2$ is $\pi/2$. Interpret this geometrically as well.

19. Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if $\left(\frac{z_1}{z_2}\right)$ is purely imaginary.

## 1.2 DE'MOIVRE'S THEOREM

### 1.2.1 INTRODUCTION

We know $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots\ldots$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots\ldots$$

$$\therefore \quad \cos x + i\sin x = 1 + ix + \frac{(ix)^2}{\underline{|2}} + \frac{(ix)^3}{\underline{|3}} + \frac{(ix)^4}{\underline{|4}} + \ldots\ldots = e^{ix}$$

So we can write $e^{\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$, in short, where

$$e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

### 1.2.2 THEOREM For all **real values** of n and $\theta$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(we will prove here only for rational values of n)

**PROOF** The expression $\cos\theta + i\sin\theta$ is abbreviated into $\text{cis}\,\theta$.

**CASE I** Let n be a positive integer.

By actual multiplication

$$\text{cis}\,\theta_1 \,.\, \text{cis}\,\theta_2 = e^{i\theta_1} \,.\, e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} = \text{cis}\,(\theta_1 + \theta_2)$$

Again $\text{cis}\,\theta_1 \,.\, \text{cis}\,\theta_2 \,.\, \text{cis}\,\theta_3 = e^{i\theta_1} \,.\, e^{i\theta_2} \,.\, e^{i\theta_3} = e^{i(\theta_1 + \theta_2 + \theta_3)}$

$$= \text{cis}(\theta_1 + \theta_2 + \theta_3)$$

Proceeding in this way n times,

$$\text{cis}\,\theta_1 \cdot \text{cis}\,\theta_2 \cdots \text{cis}\,\theta_n = \text{cis}(\theta_1 + \theta_2 + \cdots + \theta_n)$$

Now suppose $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_n = \theta$,

Then the relation becomes

$$(\text{cis}\,\theta)^n = \text{cis}\,n\theta$$

**CASE II** When n is –ve integer say $n = -m$, m is +ve integer

$$(\text{cis}\,\theta)^n = (\text{cis}\,\theta)^{-m} = \frac{1}{(\text{cis}\,\theta)^m} = \frac{1}{\text{cis}\,m\theta} = \frac{1}{\cos m\theta + i \sin m\theta}$$

$$= \frac{\cos m\theta - i \sin m\theta}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i \sin m\theta$$

Now write $m = -n$

$$= \cos(-n\theta) - i \sin(-n\theta) = \cos(n\theta) + i \sin(n\theta) = \text{cis}\,n\theta$$

Hence $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$

**CASE III** When $n = 0$, it is trivial

$$(\cos\theta + i \sin\theta)^0 = \cos 0\theta + i \sin 0\theta = 1$$

**CASE IV** When n is a fraction.

Suppose $n = \frac{p}{q}$, where p and q are integers, and q +ve.

We have $\left(\text{cis}\,\frac{\theta}{q}\right)^q = \text{cis}\,\theta$ [By case I]

$\therefore$ one value of $(\text{cis }\theta)^{1/q}$ is $\text{cis }\theta/q$

Raising to the power p we have

$$[\text{cis }\theta)^{p/q} = \left[\text{cis }\frac{\theta}{q}\right]^p = \text{cis}\left(p\,\frac{\theta}{q}\right)$$

$\therefore$ $(\text{cis }\theta)^n = \text{cis } n\theta$ , when n is fraction.

This completes the proof for n being any rational number.

**EXAMPLE 1** If $u + iv = (x_1 + iy_1)(x_2 + iy_2) \ldots\ldots (x_n + iy_n)$, prove that

$$\tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_2}{x_2}\right) + \ldots\ldots + \tan^{-1}\left(\frac{y_n}{x_n}\right)$$

**SOLUTION** Let $u + iv = R(\cos\phi + i\sin\phi) \Rightarrow \varphi = \tan^{-1}\left(\frac{v}{u}\right)$

$$x_j + iy_j = r_j(\cos\theta_j + i\sin\theta_j), \quad \text{for } j = 1, 2, 3, \ldots\ldots, n$$

$$\Rightarrow \quad \theta_j = \tan^{-1}\left(\frac{y_j}{x_j}\right)$$

Then $R\,\text{cis}(\varphi) = r_1 r_2 r_3 \ldots\ldots r_n \,\text{cis}(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n)$

$\therefore$ $R\cos\varphi = (r_1 r_2 r_3 \ldots\ldots r_n)\cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)$

$R\sin\varphi = (r_1 r_2 r_3 \ldots\ldots r_n)\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)$

$\therefore$ $\tan\varphi = \tan(\theta_1 + \theta_2 + \theta_3 \ldots\ldots + \theta_n)$

$\therefore$ $\varphi = \theta_1 + \theta_2 + \theta_3 \ldots\ldots + \theta_n$

$$\tan^{-1}\frac{v}{u} = \tan^{-1}\frac{y_1}{x_1} + \tan^{-1}\frac{y_2}{x_2} + \ldots\ldots\ldots + \tan^{-1}\frac{y_n}{x_n}$$

## EXERCISE 1.2

**1.** If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, show that one of the values of

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} \text{ is } 2\cos(m\theta - n\theta).$$

**2.** If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, prove that one of

the values of $x^m y^n + \frac{1}{x^m y^n}$ is $2\cos(m\theta + n\varphi)$.

**3.** If $2\cos\theta_j = x_j + \frac{1}{x_j}$, $j = 1, 2, 3, 4, \ldots\ldots, n$

Prove that $2\cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) = x_1x_2x_3 \ldots x_n + \frac{1}{x_1x_2x_3 \ldots x_n}$

**4.** If u, v be the roots of the equation

$z^2 - 2z + 4 = 0$, prove that

$u^n + v^n = 2^{n+1} \cos\left(\frac{n\pi}{3}\right)$

**5.** If $z_n = \cos\left(\frac{\pi}{2^n}\right) + i \sin\frac{\pi}{2^n}$, prove that

$\lim_{n \to \infty} \{z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n\} = \cos\pi$

**6.** Prove that

$(x + iy)^{\frac{m}{n}} + (x - iy)^{\frac{m}{n}} = 2(x^2 + y^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)$

and deduce $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}$

**7.** If $x_n + iy_n = (1 + i)^n$, where n is a +ve integer, prove that

(a) $x_{2n}^2 + y_{2n}^2 = 4^n$,

(b) $x_n y_{n+1} - x_{n+1} y_n = 2^n$

In questions 8 – 12 use D'eMoivre's theorem to prove the given result.

**8.** $\cos^3\theta + \sin^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta - \sin 3\theta + 3\sin\theta)$

**9.** $\cos^4\theta + \sin^4\theta = \frac{1}{4}[\cos 4\theta + 3]$

**10.** $\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$.

**11.** $[\text{cis}\,\theta - \text{cis}\,\varphi]^n + [\text{cis}\,(-\theta) - \text{cis}\,(-\varphi)]^n$

$= 2^{n+1} \sin^n\left(\frac{\theta - \varphi}{2}\right) \cos n\left(\frac{\theta + \varphi + \pi}{2}\right)$

12. $\left[\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right]^n = \text{cis}\,[n(\pi/2 - \theta)]$

where n is any integer.

13. Prove that $(\sin\theta + i\cos\theta)^n = \text{cis}\,[n(\pi/2 - \theta)]$

where n is any integer.

14. Evaluate $\left(\dfrac{\sqrt{3} - i}{\sqrt{3} + i}\right)^6$.

## 1.3 TO FIND THE NTH ROOTS OF A COMPLEX NUMBER

Let the number be $x + iy$.

Suppose $x + iy = r(\cos\theta + i\sin\theta)$, then

$x = r\cos\theta$

$y = r\sin\theta$

$\therefore \quad r = (x^2 + y^2)^{1/2}, \tan\theta = \dfrac{y}{x}$

Consequently $(x + iy)^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n}$,

$= r^{1/n}[\cos(\theta + 2p\pi) + i\sin(\theta + 2p\pi)]^{1/n}$,

where p is an integer.

Thus the n values of the nth root, i.e., $(x + iy)^{1/n}$, are

$$r^{1/n}\left[\cos\left(\frac{\theta + 2p\pi}{n}\right) + i\sin\left(\frac{\theta + 2p\pi}{n}\right)\right]$$

where $p = 0, 1, 2, 3, \ldots, n - 1$.

If p be assigned integral values $> n - 1$, these values are repeated, and no fresh root is obtained. It must be observed that nth roots of $(x + iy)$ all lie on a circle with center at origin and having radius equal to $(x^2 + y^2)^{\frac{1}{n}}$. One of the nth root has amplitude $\dfrac{\theta}{n}$ where $\theta = \tan^{-1}(y/x)$, and the others are equally spaced around the circumference of the circle.

### 1.3.1 THE VALUE OF $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$, WHERE p AND q ARE INTEGERS PRIME TO EACH OTHER AND q IS +VE.

Let k be any integer, then

$$\left[\cos\left(\frac{\theta + 2p\pi}{n}\right) + i\sin\left(\frac{\theta + 2p\pi}{n}\right)\right]^q$$

$$= \cos(p\theta + 2\pi k) + i\sin(p\theta + 2\pi k)$$

$$= \cos(p\theta) + i\sin(p\theta) = (\cos\theta + i\sin\theta)^p$$

It follows, therefore, that $\text{cis}\left(\frac{p\theta + 2\pi k}{q}\right)$ is the general value of $(\cos\theta + i\sin\theta)^{p/q}$.

Giving to k the values 0, 1, 2, 3, ....., $\overline{q-1}$, we observe that each of the q values

$$\text{cis}\left(\frac{p\theta}{q}\right), \text{cis}\left(\frac{p\theta + 2\pi}{q}\right), \text{cis}\left(\frac{p\theta + 4\pi}{q}\right), \ldots, \text{cis}\left(\frac{p\theta + 2\pi \cdot \overline{q-1}}{q}\right)$$

is a value of $(\text{cis}\,\theta)^{p/q}$. These q values are all distinct.

**EXAMPLE** Find the fourth roots of $-2 - 2i\sqrt{3}$

**SOLUTION** Here $|-2-2i\sqrt{3}| = \sqrt{4+12} = 4$

$$x < 0 \quad , \quad y < 0$$

$$\therefore \quad \text{Arg}(z) = -\pi + \tan^{-1}\frac{y}{x} = -\pi + \tan^{-1}\sqrt{3} = -\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$$

$$\therefore \quad -2 - 2i\sqrt{3} = 4\,\text{cis}\left(-\frac{2\pi}{3}\right) = 4\,\text{cis}\left(-\frac{2\pi}{3} + 2k\pi\right)$$

where k is an integer.

$$\left(-2 - 2i\sqrt{3}\right)^{\frac{1}{4}} = (4)^{1/4}\left[\text{cis}\left(-\frac{2\pi}{3} + 2k\pi\right)\right]^{1/4}$$

$$= 2^{\frac{1}{2}}\,\text{cis}\left(-\frac{\pi}{6} + \frac{k\pi}{2}\right)$$

where $k = 0, 1, 2, 3.$

$\therefore$ roots are $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right), \sqrt{2}\,\text{cis}\left(\frac{\pi}{3}\right), \sqrt{2}\,\text{cis}\,\frac{5\pi}{6}, \sqrt{2}\,\text{cis}\,\frac{4\pi}{3}$

## EXERCISE 1.3

Simplify the following:-

1. $\dfrac{(\cos 2\theta - i \sin 2\theta)^5 (\cos \theta + i \sin \theta)^{11}}{(\cos 3\theta - i \sin 3\theta)^{11} (\cos 2\theta + i \sin 2\theta)^7}$

2. $(1 - \cos \theta + i \sin \theta)^8$ 3. $(1 + \cos \theta + i \sin \theta)^{10}$

Solve the Equations.

4. $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta) = 1$

5. $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \ldots (\cos p\theta + i \sin p\theta) = 1$

where p is any +ve integer.

6. If $(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = i$

then prove that $\alpha + \beta = (4k + 1)\pi/2$, where k is an integer.

7. If $x = \cos \theta + i \sin \theta$, show that

(i) $x^n + \dfrac{1}{x^n} = 2 \cos n\theta.$

(ii) $x^n - \dfrac{1}{x^n} = 2i \sin n\theta.$

(iii) $\dfrac{x^2 - 1}{x^2 + 1} = i \tan \theta.$

(iv) $\dfrac{x^{2n} - 1}{x^{2n} + 1} = i \tan n\theta.$

8. Apply Demoivre's theorem in order to find cube roots of 1 and –1.

9. Find cube roots of $1 + i$. 10. Find all the values of $(1 - i)^{1/4}$.

11. Simplify $\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right)^{1/4}$ and express the results in a form free from trigonometric expressions.

12. If $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$,

prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma).$

13. Find the continued product of the five values of $(\cos \theta + i \sin \theta)^{3/5}$

14. Prove that $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n = \text{cis}(n\theta)$

15. Prove that

$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ has the value $-1$, if $n = 3k\pm1$ ; and the value 2, if $n = 3k$, where k is an integer.

16. Solve the following equations :-

(i) $x^7+1 = 0$ (ii) $x^7+x^4+x^3+1 = 0$

(iii) $x^{12}-1 = 0$ (iv) $(1+x)^5+x^5 = 0$

(v) $(1+x)^6+x^6 = 0$ (vi) $(5+x)^5+(5-x)^5 = 0$

## 1.4 SUM FORMULAE FOR CIRCULAR FUNCTIONS

We know

$$\cos(\theta_1+\theta_2+\cdots\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots\cdots+\theta_n)$$

$$= \cos(\Sigma\theta_1)+i\sin(\Sigma\theta_1)$$

$$= \text{cis}(\theta_1+\theta_2+\cdots\cdots+\theta_n)$$

$$= \text{cis}(\theta_1)\,\text{cis}(\theta_2)\cdots\cdots\text{cis}(\theta_n)$$

$$= (\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\cdots\cdots(\cos\theta_n+i\sin\theta_n)$$

$$= (\cos\theta_1\cdot\cos\theta_2\cdots\cdots\cos\theta_n)$$

$$\times\left(1+i\frac{\sin\theta_1}{\cos\theta_1}\right)\cdot\left(1+i\frac{\sin\theta_2}{\cos\theta_2}\right)\cdots\left(1+i\frac{\sin\theta_n}{\cos\theta_n}\right)$$

$$= (\cos\theta_1\cdot\cos\theta_2\cdots\cdots\cos\theta_n)(1+i\tan\theta_1)(1+i\tan\theta_2)\cdots\cdots(1+i\tan\theta_n)$$

$$= \cos(\theta_1)\cdot\cos(\theta_2)\cdot\cos(\theta_3)\cdots\cdots\cos(\theta_n)\times\{1+i\Sigma\tan\theta_1-\Sigma(\tan\theta_1\tan\theta_2)$$

$$-i\Sigma(\tan\theta_1\tan\theta_2\tan\theta_3)+\Sigma\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4+\cdots\cdots(i)^n\Sigma\cdots\cdots\}.$$

Writing $t_1 = \tan\theta_1+\tan\theta_2+\tan\theta_3+\cdots\cdots$

$t_2 = \tan\theta_1\tan\theta_2+\tan\theta_1\tan\theta_3+\tan\theta_2\tan\theta_3+\cdots = \Sigma\tan\theta_1\tan\theta_2$

$t_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3\cdots$ etc.

$$= \prod_{j=1}^{n} \cos\theta_j \{1 + it_1 + i^2t_2 + i^3t_3 + i^4t_4 + \cdots + i^nt_n\}.$$

We know $i^2 = -1$, $i^4 = 1$, $i^6 = -1$, .....

$i^3 = -i$, $i^5 = i$, .....

From (i) and (ii) equating real and imaginary part

$$\cos(\Sigma\theta_1) = \prod_{j=1}^{n} \cos\theta_j \{1 - t_2 + t_4 - t_6 + t_8 + \ldots\ldots\}$$

$$\sin(\Sigma\theta_1) = \prod_{j=1}^{n} \cos\theta_j \{t_1 - t_3 + t_5 - t_7 + \ldots\ldots\}$$

Dividing

$$\tan(\Sigma\theta_1) = \frac{t_1 - t_3 + t_5 - t_7 + \cdots\cdots}{1 - t_2 + t_4 - t_6 + t_8 \cdots\cdots}$$

## 1.4.1 EXPRESSION OF COS nθ, SIN nθ, IN TERMS OF COS θ AND SIN θ, n BEING A +VE INTEGER.

Denoting sin θ, cos θ, tan θ by s, c, t respectively.

**CASE I** When $n = 2m$ (even positive integer),

$$(\cos\theta + i\sin\theta)^n = \text{cis}\, n\theta = \cos n\theta + i\sin n\theta \quad \ldots\ldots \text{(i)}$$

$$(c + is)^n = (c + is)^{2m} = c^{2m} + \binom{2m}{1} c^{2m-1}(is)^1 + \binom{2m}{2} c^{2m-2}(is)^2 + \ldots$$

$$\ldots + \binom{2m}{2m-1} c\,(is)^{2m-1} + \binom{2m}{2m}(is)^{2m} \quad \ldots\ldots \text{(ii)}$$

From (i) and (ii) equating real and imaginary parts.

$$\cos n\theta = c^{2m} - \binom{2m}{2} c^{2m-2}s^2 + \ldots\ldots + (-1)^m s^{2m}$$

$$\sin n\theta = \binom{2m}{1} c^{2m-1}s - \binom{2m}{3} c^{2m-3}s^3 + \ldots\ldots + (-1)^{2m-1}\binom{2m}{2m-1} c\, s^{2m-1}$$

$$\therefore \tan n\theta = \frac{\binom{2m}{1} c^{2m-1} s - \binom{2m}{3} c^{2m-3} s^3 + \ldots\ldots}{c^{2m} - \binom{2m}{2} c^{2m-2} s^2 + \ldots\ldots}$$

$$= \frac{\binom{2m}{1} t - \binom{2m}{3} t^3 + \ldots\ldots}{1 - \binom{2m}{2} t^2 + \binom{2m}{4} t^4 - \ldots\ldots\ldots}$$

**CASE II** Similarly when n is odd. Say $n = 2m + 1$

$$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{2m+1} = \cos(2m+1)\theta + i \sin(2m+1)\theta$$

$$= (c+i\,s)^{2m+1}$$

$$= c^{2m+1} + \binom{2m+1}{1} c^{2m}(i\,s) + \binom{2m+1}{2} c^{2m-1}(i\,s)^2 + \cdots + \binom{2m+1}{2m} c(i\,s)^{2m} + \binom{2m+1}{2m+1} (i\,s)^{2m+1}$$

Hence equating real and imaginary parts,

$$\cos(2m+1)\theta = c^{2m+1} - \binom{2m+1}{2} c^{2m-1} s^2 + \binom{2m+1}{4} c^{2m-3} s^4 + \ldots\ldots + (-1)^m \binom{2m+1}{2m} c\, s^{2m}$$

$$\sin(2m+1)\theta = \binom{2m+1}{1} c^{2m} s - \binom{2m+1}{3} c^{2m-2} s^3 + \ldots\ldots + (-1)^m \binom{2m+1}{2m+1} s^{2m+1}$$

Dividing we get

$$\tan(2m+1)\theta = \frac{\binom{2m+1}{1} t - \binom{2m+1}{3} t^3 + \binom{2m+1}{5} t^5 \ldots\ldots}{1 - \binom{2m+1}{2} t^2 + \binom{2m+1}{4} t^4 \ldots\ldots\ldots}$$

**EXAMPLE** Express $\cos 5\theta$, $\sin 5\theta$ and $\tan 5\theta$ in terms of $\cos\theta$, $\sin\theta$ and $\tan\theta$.

**SOLUTION** Let c, s and t stands for $\cos\theta$, $\sin\theta$ and $\tan\theta$ respectively.

Then $\cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5$

$$= (c + is)^5 = c^5 + \binom{5}{1} c^4 (i\,s) + \binom{5}{2} c^3 (i\,s)^2 + \binom{5}{3} c^2 (i\,s)^3 + \binom{5}{4} c\,(i\,s)^4 + \binom{5}{5} (i\,s)^5$$

Equating real and imaginary parts.

$$\cos 5\theta = c^5 - \binom{5}{2} c^3 s^2 + \binom{5}{4} c\, s^4$$

$$= c^5 - \frac{\lfloor 5}{\lfloor 2 \lfloor 3} c^3 (1 - c^2) + \frac{\lfloor 5}{\lfloor 4 \cdot \lfloor 1} c (1 - c^2)^2$$

$$= c^5 - 10c^3 (1 - c^2) + 5c (1 - 2c^2 + c^4)$$

$$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$$

$$= 16c^5 - 20c^3 + 5c$$

$$\therefore \cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

$$\sin 5\theta = \binom{5}{1} c^4 s - \binom{5}{3} c^2 s^3 + \binom{5}{5} s^5$$

$$= 5(1 - s^2)^2 s - 10 (1 - s^2) s^3 + s^5$$

$$= 5s(1 - 2s^2 + s^4) - 10s^3 (1 - s^2) + s^5$$

$$= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$$

$$= 16s^5 - 20s^3 + 5s = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

Dividing $\sin 5\theta$ by $\cos 5\theta$

$$\tan 5\theta = \frac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5c\, s^4}$$

Dividing numerator and donominator by $c^5$

$$\tan 5\theta = \frac{5 \frac{s}{c} - 10 \left(\frac{s}{c}\right)^3 + \left(\frac{s}{c}\right)^5}{1 - 10 \left(\frac{s}{c}\right)^2 + 5 \left(\frac{s}{c}\right)^4} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

$$= \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

## EXERCISE 1.4

1. Write the value of cos 7θ, sin 7θ and tan 7θ in terms of cos θ, sin θ and tan θ respectively.

   If c and s stand for cos θ and sin θ respectively, prove the following:

2. $\sin 6\theta = 6c^5s - 20c^3s^3 + 6cs^5$

3. $\cos 8\theta = c^8 - 28c^6s^2 + 70c^4s^4 - 28c^2s^6 + s^8$

4. $\cos 9\theta \sec\theta = c^8 - 36c^6s^2 + 126c^4s^4 - 84c^2s^6 + 9s^8$

5. $\sin 7\theta \operatorname{cosec}\theta = 128c^7 - 192c^5 + 80c^3 - 8c$

6. $\cos(8\theta) = 128c^8 - 256c^6 + 160c^4 - 32c^2 + 1$

7. When n is even +ve integer, prove that the last term in the expansions of sin nθ and cos nθ is respectively,

   $$n(-1)^{\frac{n-2}{2}} c\, s^{n-1} \quad \text{and} \quad (-1)^{n/2} s^n,$$

   where $c \equiv \cos\theta$ and $s \equiv \sin\theta$

8. Prove that

   $$1 - \frac{n(n-1)}{\underline{|2}} + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} - \ldots\ldots = 2^{n/2}\cos\frac{n\pi}{4},$$

   n being a +ve integer.

9. Prove that

   $$n - \frac{n(n-1)(n-2)}{\underline{|3}} + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|4}} - \ldots\ldots$$

   $= 2^{n/2} \sin n\pi/4$, n being a +ve integer.

10. Prove that $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{8\pi}{7}$ are the roots of the equation $8x^3 + 4x^2 - 4x - 1 = 0$ and deduce that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} - \frac{1}{2} = 0$.

## 1.5. TO EXPRESS $\cos^n\theta$ IN TERMS OF COSINES OF MULTIPLE OF $\theta$, n BEING ANY +VE INTEGER

Let $x = \text{cis}\,\theta = e^{i\theta} \Rightarrow x^n = \text{cis}\,(n\theta)$

$\therefore \frac{1}{x} = \text{cis}\,(-\theta) = e^{-i\theta} \Rightarrow \frac{1}{x^n} = \text{cis}\,(-n\theta)$

$x + \frac{1}{x} = \text{cis}\,(\theta) + \text{cis}\,(-\theta) = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$

Similarly $x - \frac{1}{x} = 2i\sin\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$

$x^n - \frac{1}{x^n} = 2i\sin n\theta$

Hence $\left(x + \frac{1}{x}\right)^n = (2\cos\theta)^n = 2^n\cos^n\theta$

Expanding $\left(x + \frac{1}{x}\right)^n$ by Binomial Theorem.

$$2^n\cos^n\theta = x^n + \binom{n}{1}x^{n-1}\cdot\frac{1}{x} + \binom{n}{2}x^{n-2}\cdot\frac{1}{x^2} + \cdots\cdots + \binom{n}{n}\frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) + \binom{n}{1}\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots\cdots$$

Because $\binom{n}{r} = \binom{n}{n-r}$

$$2^n\cos^n\theta = 2\cos n\theta + \binom{n}{1}2\cos(n-2)\theta + \binom{n}{2}2\cos(n-4)\theta + \cdots\cdots$$

$$\therefore 2^{n-1}\cos^n\theta = \cos n\theta + \binom{n}{1}\cos(n-2)\theta + \binom{n}{2}\cos(n-4)\theta + \cdots\cdots$$

If n is odd, the last term will be $\binom{n}{\frac{n-1}{2}} \cos\theta$; and if n is even, the last term will be $\frac{1}{2}\binom{n}{n/2}$.

## 1.5.1 TO EXPRESS $\sin^n\theta$ IN TERMS OF SINES OR COSINES OF MULTIPLES OF $\theta$, ACCORDING AS n IS ODD OR EVEN POSITIVE INTEGER.

As in the previous article,

$$(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$$

$$= x^n - \binom{n}{1}x^{n-1}\cdot\frac{1}{x} + \binom{n}{2}x^{n-2}\cdot\frac{1}{x^2} + \cdots\cdots - \binom{n}{n-1}x\cdot\frac{1}{x^{n-1}} + \binom{n}{n}\frac{1}{x^n}$$

, when n is even.

As n is even integer, $\therefore\ (i)^n = (-1)^{n/2}$

$$(-1)^{n/2}\,2^n\sin^n\theta = \left(x^n + \frac{1}{x^n}\right) - \binom{n}{1}\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right)\cdots$$

$$(-1)^{n/2}\,2^n\sin^n\theta = 2\cos n\theta - 2\binom{n}{1}\cos(n-2)\theta + 2\binom{n}{2}\cos(n-4)\theta + \cdots$$

$$(-1)^{n/2}\,2^{n-1}\sin^n\theta = \cos n\theta - \binom{n}{1}\cos(n-2)\theta + \binom{n}{2}\cos(n-4)\theta\ \cdots$$

If n is odd,

$$(2i\sin\theta)^n = x^n - \binom{n}{1}x^{n-1}\cdot\frac{1}{x} + \binom{n}{2}x^{n-2}\cdot\frac{1}{x^2}\cdots + \binom{n}{n}\left(\frac{-1}{x}\right)^n$$

$$= \left(x^n - \frac{1}{x^n}\right) - \binom{n}{1}\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right)\cdots\cdots$$

$$(2i\sin\theta)^n = 2i\sin n\theta - \binom{n}{1}2i\sin(n-2)\theta + \binom{n}{2}2i\sin(n-4)\theta\cdots\cdots$$

$$(2i)^{n-1}\sin^n\theta = \sin n\theta - \binom{n}{1}\sin(n-2)\theta + \binom{n}{2}\sin(n-4)\theta\cdots\cdots$$

The last term being $(-1)^{\frac{n-1}{2}} \binom{n}{\frac{n-1}{2}} \sin\theta$.

**EXAMPLE** Expand $\cos^6\theta$ in terms of cosines of multiples of $\theta$.

**SOLUTION** Let $x = \cos\theta + i\sin\theta \Rightarrow x^n = \text{cis}(n\theta)$

$$\frac{1}{x} = \cos\theta - i\sin\theta \Rightarrow x^{-n} = \text{cis}(-n\theta)$$

$$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6 = x^6 + \binom{6}{1}x^4 + \binom{6}{2}x^2 + \binom{6}{3}x^0 + \binom{6}{4}\cdot\frac{1}{x^2} + \binom{6}{5}\frac{1}{x^4} + \binom{6}{6}\frac{1}{x^6}.$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

$$2^6\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$\cos^6\theta = \frac{1}{2^5}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$$

## EXERCISE 1.5

Prove the following:

1. $\cos^7\theta = \frac{1}{2^6}\{\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\}$

2. $\sin^7\theta = \frac{-1}{2^6}[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\}$

3. $64\cos^3\theta\sin^4\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta.$

Hint. $(2\cos\theta)^3(2i\sin\theta)^4 = \left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^4$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$$

$$= \left[x^6 - 3\left(x^2 - \frac{1}{x^2}\right) - \frac{1}{x^6}\right]\left[x - \frac{1}{x}\right]$$

$$= \quad x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \quad \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$\therefore \quad 2^7 \cos^3\theta \sin^4\theta = 2\cos 7\theta - 2\cos 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)$

4. $32 \sin^4\theta \cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

5. $2^{11} \sin^5\theta \cos^7\theta = \sin 12\theta + 2\sin 10\theta - 4\sin 8\theta - 10\sin 6\theta + 5\sin 4\theta + 20\sin 2\theta$

6. $2^9 \cos^7\theta \sin^2\theta = -\sin 10\theta - 4\sin 8\theta - 3\sin 6\theta + 8\sin 4\theta + 14\sin 2\theta.$

7. $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35.$

## 1.6 THE EXPONENTIAL FUNCTION

The exponential function of complex number is defined by the equation,

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

i.e. $$e^z = e^x(\cos y + i \sin y)$$

While evaluating we agree that y is to be taken in radian.

When $y = 0$ i.e., z is purely real,

$e^z = e^x$

When z is purely imaginary number $i\theta$,

$e^z = e^{i\theta} = \cos\theta + i\sin\theta$

which was introduced earlier is known as Euler's formula.

Writing $e^x = \rho > 0$, and $y = \varphi$,

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$= \rho(\cos\phi + i \sin\varphi).$$

so that $|e^z| = \rho$, $\text{Arg}(e^z) = \varphi$

By the transformation,

$$W = ez = \rho(\cos\varphi + i \sin\varphi),$$

W is the image of $z = \log_e \rho + i\varphi$

i.e.

$$W = e^{\log \rho + i\varphi} = e^{\log\rho} \cdot e^{i\varphi}$$

$$= \rho e^{i\varphi} = \rho (\cos \varphi + i \sin \varphi)$$

$W = e^z \neq 0$ for any complex number z.

This fact can be stated in this manner also, that the range of the exponential function $e^z$ is the whole complex plane except origin.

$$e^{z+2\pi i} = e^z$$

That is, $e^z$ is periodic with a pure imaginary period of $2\pi i$. Moreover, it can be verified that $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

and $e^{z_1} = e^{z_2} \Rightarrow z_1 = z_2 + 2k\pi i$, where k is any integer.

If *a* is real and +ve, we define $a^z = e^{\log_e a^z} = e^{z \log_e a}$.

$$(e^z)^n = e^{nz} \quad (n = 1, 2, 3, 4, \ldots\ldots)$$

**EXAMPLE** · Solve for $z, e^z = -1$.

**SOLUTION**·Here suppose $e^z = -1 = \rho (\cos \varphi + i \sin \varphi)$

$$\rho \cos \varphi = -1$$

$$\rho \sin \varphi = 0$$

$$\rho^2 = 1 + 0 \Rightarrow \rho = 1$$

$$\varphi = \pi + 2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

$$z = \log\rho + i\varphi$$

$$\therefore \quad z = \log 1 + i(\pi + 2n\pi)$$

$$z = 0 + \pi i (1 + 2n), \text{ where n is any integer.}$$

## EXERCISE 1.6

1. Show that

   (i) $|e^{iZ}| = 1$

   (ii) $e^{Z} = 1$ if and only if $z = 2k\pi i$, where k is any integer.

   (iii) $|e^{Z}| = e^{x}$

2. Show that

   (i) $e^{(2\pm 3\pi i)} = -e^2$

   (ii) $e^{\frac{1}{2}+\frac{\pi}{4}i} = \sqrt{e}\left(\frac{1+i}{\sqrt{2}}\right)$

   (iii) $e^{Z+\pi i} = -e^{z}$

   (iv) $e^{8\pi i} = 1$

3. Find all values of z such that

   (i) $e^{Z} = -2$,

   (ii) $e^{Z} = 1 + \sqrt{3}\,i$.

   (iii) $e^{2Z-1} = 1$

4. Write $|e^{2Z+i}|$ and $|e^{iZ^2}|$ in terms of x and y

5. Prove that $|e^{-2Z}| < 1$ if and only if $\mathrm{Re}(z) > 0$

6. If $z \neq 0$, show that if $z = re^{i\theta}$, then

   (i) $\bar{z} = re^{-i\theta}$,

   (ii) $e^{\log r + i\theta} = z$

7. Show that $e^{-nZ} = \left(\frac{1}{e^{Z}}\right)^{n}$ for $n = 1, 2, 3, 4, \ldots\ldots$

8. Show that $e^{\bar{z}} = \overline{(e^{Z})}$ for all z.

9. Show that $e^{i\bar{z}} = \overline{(e^{iZ})}$ if and only if $z = n\pi$, where

   $n = 0, \pm 1, \pm 2, \ldots\ldots$

**10.** Examine the behavior of

(i) $e^{x+iy}$ as $x \to -\infty$

(ii) $e^{x+iy}$ as $y \to \infty$

## 1.7 TRIGONOMETRIC FUNCTIONS OR CIRCULAR FUNCTIONS

We know $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

By addition and subtraction we get

$$2\cos\theta = e^{i\theta} + e^{-i\theta}, \quad 2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

which results in

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

for every real number $\theta$.

In the same manner sine and cosine functions of the complex variable z can be defined as follows.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

The other four trigonometric functions are defined in terms of the sine and cosine functions by the usual relations:-

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}$$

$$\operatorname{cosec} z = \frac{1}{\sin z}$$

### 1.7.1 HYPERBOLIC FUNCTIONS

The three fundamental operations of Arithmetic on the functions $e^x$, and $e^{-x}$ give new functions which are called Hyperbolic functions and are defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{sech } x = \frac{2}{e^x + e^{-x}}, \quad \text{cosech } x = \frac{2}{e^x - e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

These are called, respectively, hyperbolic cosine, hyperbolic sine etc.

**NOTE:**

Since $\frac{e^x + e^{-x}}{2}$ and $\frac{e^x - e^{-x}}{2}$ turn out to be geometrically describable as the X – and Y – projections of points on a hyperbola $X^2 - Y^2 = 1$, so they are called hyperbolic functions.

The above functions can similarly be defined for a complex variable z as under:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \text{sech } z = \frac{1}{\cosh z}$$

$$\coth z = \frac{\cosh z}{\sinh z}, \quad \text{cosech } z = \frac{1}{\sinh z}$$

## 1.7.2 THE RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTION

$$\sinh (i z) = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$$

$$\cosh (i z) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\text{and } \sin (i z) = \frac{e^{i(iZ)} - e^{-i(iZ)}}{2i} = i\left(\frac{e^z - e^{-z}}{2}\right) = i \sinh z$$

$$\cos (i z) = \frac{e^{i(iZ)} + e^{-i(iZ)}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$$

## 1.7.3 FUNDAMENTAL RELATIONS BETWEEN THE HYPERBOLIC FUNCTIONS OF THE SAME ARGUMENT

$$\cosh^2 z - \sinh^2 z = 1 \quad \ldots\ldots (1)$$

$$\text{sech}^2 z + \tanh^2 z = 1 \quad \ldots\ldots (2)$$

$$\coth^2 z - \text{cosech}^2 z = 1 \quad \ldots\ldots (3)$$

In order to prove (1) we start as follows:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

By addition and subtraction, respectively,

$$\cosh z + \sinh z = e^z$$

$$\cosh z - \sinh z = e^{-z}$$

$$\therefore \quad (\cosh z - \sinh z)(\cosh z + \sinh z) = e^z \cdot e^{-z}$$

$$\cosh^2 z - \sinh^2 z = e^0 = 1$$

Dividing both sides of (1) by $\cosh^2 z$ we get

$$1 - \tanh^2 z = \operatorname{sech}^2 z$$

$$1 = \tanh^2 z + \operatorname{sech}^2 z \quad \text{which is (2).}$$

Dividing (1) by $\sinh^2 z$ we get

$$\coth^2 z - 1 = \operatorname{cosech}^2 z$$

$$\coth^2 z - \operatorname{cosech}^2 z = 1 \quad \text{which is (3).}$$

**EXAMPLE 1** Prove $\sin z = \sin x \cosh y + i \cos x \sinh y$.

**PROOF** $\sin z = \sin (x + iy) = \sin x \cos (iy) + \cos x \sin (iy)$

But $\cos (iy) = \cosh y$, $\sin (iy) = i \sinh y$.

$$\therefore \quad \sin z = \sin x \cosh y + i \cos x \sinh y.$$

Similarly,

$$\cos z = \cos (x + iy) = \cos x \cos (iy) - \sin x \sin (iy)$$
$$= \cos x \cosh y - i \sin x \sinh y.$$

**EXAMPLE 2** Prove that $\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

**SOLUTION** We know $\sin (i z) = i \sinh z$

$$\therefore \quad \sinh z = \frac{1}{i} \sin (i z) = -i \sin (i z)$$

Hence $\sinh (z_1 + z_2) = -i \sin (i z_1 + i z_2)$

$$= -i\{\sin(i z_1)\cos(i z_2) + \cos(i z_1)\sin(i z_2)\}$$

$$= -i\{i\sinh z_1 \cosh z_2 + i\cosh z_1 \sinh z_2\}, \quad (i^2 = -1)$$

$$\therefore \quad \sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

## 1.7.4 INVERSE CIRCULAR FUNCTIONS

### DEFINITION

If $z = \sin w$, then $w = \sin^{-1} z$ is called the inverse sine function of z. In the same way the other inverse circular functions are defined. These functions can be written in the form of natural logarithms as follows:-

$$w = \sin^{-1} z \Rightarrow z = \sin w$$

$$\therefore \quad z = \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - e^{-iw}$$

$$(e^{iw})^2 - (2iz)e^{iw} - 1 = 0$$

$$e^{iw} = \frac{2iz \pm \sqrt{4i^2 z^2 + 4}}{2} = iz \pm \sqrt{1 - z^2} \qquad \ldots\ldots (1)$$

where $\sqrt{1 - z^2}$ takes two values for each $z \neq 1$. Moreover $-\sqrt{1 - z^2}$ assumes the same values as $\sqrt{1 - z^2}$. Therefore, we can write (1) as

$$e^{iw} = iz + \sqrt{1 - z^2}$$

$$\text{or} \quad iw = \log\left(iz + \sqrt{1 - z^2}\right)$$

$$w = \frac{1}{i}\log\left(iz + \sqrt{1 - z^2}\right)$$

But $w = \sin^{-1} z$

$$\therefore \quad \sin^{-1} z = \frac{1}{i}\log\left(iz + \sqrt{1 - z^2}\right)$$

Similarly

$$\cos^{-1} z = \frac{1}{i}\log\left(z + \sqrt{z^2 - 1}\right)$$

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right);$$

$$\cot^{-1} z = \frac{1}{2i} \log\left(\frac{z + i}{z - i}\right);$$

$$\sec^{-1} z = \frac{1}{i} \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right);$$

$$\operatorname{cosec}^{-1} z = \frac{1}{i} \log\left(\frac{i + \sqrt{z^2 - 1}}{z}\right)$$

## 1.7.5 INVERSE HYPERBOLIC FUNCTIONS

### DEFINITION

If $z = \sinh w$, then w is known as inverse hyperbolic sine of z. Similarly we can define the other Inverse Hyperbolic functions. Inverse Hyperbolic functions can also be written in the form of natural logarithms as follows:-

$$w = \sinh^{-1} z \Rightarrow z = \sinh w = \frac{e^w - e^{-w}}{2}$$

or $e^w - e^{-w} = 2z$

$$(e^w)^2 - 2z(e^w) - 1 = 0$$

$$\therefore \quad e^w = \frac{2z \pm \sqrt{4z^2 + 4}}{2} = z + \sqrt{z^2 + 1},$$

omitting the –ve sign as in previous article.

$$\therefore \quad w = \log\left(z + \sqrt{z^2 + 1}\right)$$

But $w = \sin h^{-1} z$

$$\therefore \quad \sinh^{-1} z = \log\left(z + \sqrt{z^2 + 1}\right)$$

Similarly,

$$\cosh^{-1} z = \log\left(z + \sqrt{z^2 - 1}\right)$$

$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right), \coth^{-1} z = \frac{1}{2} \log\left(\frac{z + 1}{z - 1}\right).$$

$$\text{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right), \text{cosech}^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

**EXAMPLES 3** Prove that $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

**SOLUTION** Let $w = \tanh^{-1} z$

$$\therefore \quad z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}}$$

Now

$$\frac{z}{1} = \frac{e^w - e^{-w}}{e^w + e^{-w}}$$

By componendo and dividendo,

$$\frac{z+1}{z-1} = \frac{2e^w}{-2e^{-w}} \Rightarrow \frac{1+z}{1-z} = e^{2w}$$

$$\text{or} \quad 2w = \log\left(\frac{1+z}{1-z}\right)$$

$$\text{or} \quad w = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

$$\therefore \quad \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

**EXAMPLE 4** Prove that $\sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

**SOLUTION** Let $w = \sec^{-1} z \Rightarrow z = \sec w$

$$\text{or} \quad \frac{1}{z} = \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

$$\text{or} \quad 2 = z e^{iw} + z e^{-iw}$$

$$z(e^{iw})^2 - 2(e^{iw}) + z = 0$$

$$\therefore \quad e^{iw} = \frac{2 \pm \sqrt{4-4z^2}}{2z}$$

$$= \frac{1 + \sqrt{1 - z^2}}{z}, \quad \text{[leaving –ve sign]}$$

$$iw = \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$

$$\text{or} \quad w = \frac{1}{i} \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$

$$\therefore \quad \sec^{-1} z = \frac{1}{i} \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$

## EXERCISE 1.7

1. Prove that

   (i) $\sin(z + 2\pi) = \sin z$,

   (ii) $\cos(z + 2\pi) = \cos z$

   (iii) $\tan(\pi + z) = \tan z$,

   (iv) $\cos(z + \pi) = -\cos z$

   (v) $\sin(z + \pi) = -\sin z$.

   What character of sine z, cosine z and tangent z is observed from (i), (ii) and (iii)respectively?

2. Prove that

   (i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

   (ii) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

   where $z = x + iy$.

3. Prove that

   (i) $\sin^2 z + \cos^2 z = 1$

   (ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

   (iii) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

4. Prove that $\sin z = 0$ if and only if $z = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

5. Prove that $\cos z = 0$ if and only if $z = (2n + 1)\pi/2$, $(n = 0, \pm1, \pm2, \ldots)$

6. Show that for any complex number z, $e^{iz} = \cos z + i \sin z$

7. Prove that (i) $1 + \tan^2 z = \sec^2 z$ (ii) $1 + \cot^2 z = \operatorname{cosec}^2 z$.

8. Establish the identities

   (i) $2 \sin (z_1 + z_2) \sin (z_1 - z_2) = \cos (2z_2) - \cos (2z_1)$

   (ii) $2 \cos (z_1 + z_2) \sin (z_1 - z_2) = \sin (2z_1) - \sin (2z_2)$

9. Show that $\cos (i\,\bar{z}) = \overline{\cos (i\,z)}$ for all z.

10. Show that $\sin (i\,\bar{z}) = \overline{\sin (i\,z)}$ if and only if $z = n\pi i$, $(n = 0, \pm1, \pm2, \ldots)$.

11. Show that if $\cos z_1 = \cos z_2$ then either $z_1 + z_2$ or $z_1 - z_2$ is an integral multiple of $2\pi$. Also obtain a similar result for $\sin z_1 = \sin z_2$.

12. Solve the equations

   (i) $\sin z = \cosh 4$,

   (ii) $\cos z = 2$.

13. Show that $2\pi i$ is period of sinh z and cosh z while $\pi i$ is period of tanh z.

14. Solve (i) $\sinh z = 0$ (ii) $\cosh z = 0$

   (iii) $\cosh z = 1/2$; (iv) $\sinh z = i$; (v) $\cosh z = -2$

15. If $\cosh u = \sec \theta$, prove that $\sinh u = \tan \theta$, $\tanh u = \sin \theta$

16. Prove that (i) $\tan (x + iy) = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

   (ii) $\cot (x + iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$

   (iii) $\sec (x + iy) = 2\left[\dfrac{\cos x \cosh y + i \sin x \sinh y}{\cosh 2x + \cosh 2y}\right]$

17. If $\tan (x + iy) = \sin (u + iv)$, prove that

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

18. Let $P = a + ib$, $q = a - ib$, where $a$, b are real.
   Show that $pe^{p} + qe^{q}$ is real.

19. If $\cos\alpha \cosh\beta = \cos\varphi$, $\sin\alpha \sinh\beta = \sin\varphi$; prove that

$$\sin\varphi = \pm\sin^2\alpha = \pm\sinh^2\beta$$

20. If $\sin(u + iv) = x + iy$ prove that

(i) $\dfrac{x^2}{\cosh^2 v} + \dfrac{y^2}{\sinh^2 v} = 1.$

(ii) $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1.$

21. If $x + iy = \cos(u + iv)$, prove that

$$(1 + x)^2 + y^2 = (\cosh v + \cos u)^2,$$
$$(1 - x)^2 + y^2 = (\cosh v - \cos u)^2$$

22. If $\sin(\theta + i\varphi) = \cos\alpha + i\sin\alpha$, then $\cos^2\theta = \pm\sin\alpha$

23. If $\tan(u + iv) = x + iy$, prove that

$x^2 + y^2 + 2x\cot 2u = 1$ and $x^2 + y^2 - 2y\coth 2v + 1 = 0$

24. If $x + iy = c\cos h(\theta + i\varphi)$, prove that

$$x^2\operatorname{sech}^2\theta + y^2\operatorname{cosech}^2\theta = c^2,$$

and $x^2\sec^2\varphi - y^2\operatorname{cosec}^2\varphi = c^2.$

25. Separate the real and imaginary parts of $\tan^{-1}(x + iy)$.

26. If $\tan(u + iv) = \tan x + i\sec x$, prove that

$$e^{2v} = \pm\cot\frac{x}{2} \quad \text{and} \quad 2u = n\pi + \frac{\pi}{2} + x$$

where n is any integer.

## 1.8 LOGARITHMS OF COMPLEX NUMBERS

### 1.8.1 DEFINITIONS

If z and w are two complex variables such that $w = e^z$, then z is called a natural logarithm of w and we write $z = \log w$.

If $w = u + iv$ and $z = x + iy$,

then $u + iv = e^{x+iy} \Rightarrow x + iy = \log(u + iv)$

## 1.8.2 THE VALUES OF LOG W

Let $w = \rho(\cos\theta + i\sin\theta)$ and let $z = x + iy$ be a natural logarithm of w.

By definition

$$w = e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

or $$\rho(\cos\theta + i\sin\theta) = e^x[\cos(y + 2n\pi) + i\sin(y + 2n\pi)]$$

$$\therefore \rho = e^x, \quad \theta = y + 2n\pi,$$

or $$x = \log_e \rho, \quad y = \theta - 2n\pi = \theta + 2m\pi,$$

where $m = -n = 0, \pm1, \pm2, \pm3, \ldots\ldots$

Hence $$\text{Log } w = \log\rho(\cos\theta + i\sin\theta)$$

$$= x + iy = \log\rho + (\theta + 2m\pi)\,i. \quad \ldots\ldots \text{(A)}$$

Thus natural logarithm of complex numbers is a many valued function.

If $\theta_1$ is the value of $\theta$ in (A) such that

$$-\pi < \theta_1 \leq \pi$$

The principal value of Log w is defined by the equation,

$$\text{Log } w = \log\rho + i\theta_1, \quad \text{where } \rho = |w|, \quad \theta_1 = \text{Arg}(w)$$

Some values of logarithmic functions obey the following rules when the variable under discussion is a complex variable.

$$\text{Log } w_1w_2 = \text{Log } w_1 + \text{Log } w_2$$

$$\text{Log}\left(\frac{w_1}{w_2}\right) = \text{Log } w_1 - \text{Log } w_2$$

$$\text{Log}\left(\frac{1}{w}\right) = -\text{Log } w$$

$$\text{Log } (w)^n = n \log w.$$

Generally these results do not hold for principal values of logarithms.

## 1.8.3 GENERALIZED INDICES

### DEFINITION

If z and w are two complex numbers, then we define

$$z^w = e^{W \log Z}$$

Now $|z^w| = |e^{W \log Z}| = |e^{(u+iv)[\log|Z| + i\,\mathrm{Arg}(Z)]}|$

where $w = u + iv$, $|z| = r$, $\theta = \mathrm{Arg}(z)$,

$z = x + iy$

$= |e^{(u+iv)(\log r + i\theta)}| = e^{u \log r - v\theta}$

$= e^{\log r^u} \cdot e^{-v\theta} = r^u \cdot e^{-v\theta}$

Remember we are considering here principal argument of z.

Amplitude of $z^w$ or principal Argument of $z^w$ is given by,

$$\mathrm{Arg}(z^w) = \theta u + v \log r$$

Remember Again we are considering the principal logarithm as well as principal Argument of $z^w$ and z.

**EXAMPLE** Find the general value and the principal value of log (–5).

**SOLUTION** Let $x + iy = \log(-5)$

$$\therefore \quad -5 = e^{x+iy} = e^x(\cos y + i \sin y)$$

Equating real and Imaginary parts.

$$\left.\begin{aligned} -5 &= e^x \cos y \\ 0 &= e^x \sin y \end{aligned}\right\} \Rightarrow e^x = 5$$

$\therefore \quad \cos y = -1 \Rightarrow y = \pi$

$\sin y = 0 \Rightarrow y = \pi$

The general solution is $y = \pi + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots\ldots$

$\therefore \quad -5 = 5\,(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))$

$$-5 = 5\ e^{i(2K+1)\pi}$$

$$\text{Log}(-5) = \log 5 + i(2k+1)\pi$$

Principal value is given as

$$\text{Log}(-5) = \log 5 + i\pi$$

**ALTERNATIVE METHOD**

$$\text{Log}(z) = \log|z| + i \arg(z)$$

Here $z = -5$

$|z| = 5,$ $\arg(-5) = \pi + 2k\pi$

$$\therefore \quad \text{Log}(-5) = \log 5 + i[\pi + 2k\pi]$$

The principal value is given by

$$\text{Arg}(-5) = \pi$$

$$\therefore \quad \text{Log}(-5) = \log 5 + i\pi.$$

## EXERCISE 1.8

**1.** Prove that

(i) $\text{Log}(-3) = \log(3) + i(2k+1)\pi$, where k is integer.

(ii) $\text{Log}(-1) = i\pi$

(iii) $\text{Log}(i) = i\pi/2$

(iv) $\text{Log}(1+i) = \frac{1}{2}\log 2 + i\pi/4$

**2.** Find the principal value of $(i)^i$.

**3.** If x is real prove that

$$2i \tan^{-1} x = \log\left(\frac{1+ix}{1-ix}\right)$$

**4.** Show that $\log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\frac{b}{a}\right)$

**5.** *(i)* If $(\alpha + ib)p = m^{x+iy}$, prove that one of the values of $\frac{y}{x}$ is $\frac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)}$.

(ii) If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$, $a > 0$ then find $\alpha$ and $\log_a(x^2+y^2)$.

6. Separate the real and imaginary parts of $\log \sin (x + iy)$.

7. Separate the following into their real and imaginary parts.

(i) $\tan^{-1} (\cos \theta + i \sin \theta)$ (ii) $\log \cos (x + iy)$

(iii) $\tan^{-1}\left(\frac{z}{\bar{z}}\right)$, where z lies in quadrant I.

(iv) $(\alpha + i\beta)^{p+iq}$ (v) $(-1 + i)^{i+\sqrt{3}}$

8. If $\tan (u + iv) = e^{ix}$, then find u and v in terms of x.

9. If $\cosh (x + iy) = \cot (u + iv)$, prove that

$$\sinh 2v \operatorname{cosec} 2u + \tanh x \tan y = 0$$

and $$\coth 2u = -\frac{\cosh 2x + \cos 2y + 2}{4 \sinh x \sin y}$$

10. If $\cosh (u + iv) = \tan (x + iy)$, prove that

$$\cosh 2u + \cos 2v = 2\frac{(\cosh 2y - \cos 2x)}{(\cosh 2y + \cos 2x)}$$

11. If $\tan (\theta + i\varphi) = \sin (x + iy)$, then

$$\text{cothy} \sinh 2\varphi = \cot x \sin 2\theta.$$

12. Find the values of (i) $\tan^{-1} (2i)$, (ii) $\tanh^{-1} (0)$

(iii) $\cosh^{-1} (-1)$ (iv) $\tan^{-1} (1 + i)$.

## 1.9 APPLICATION OF D'MOIVRE'S THEOREM IN SUMMATION OF SERIES BY THE USE OF THE EXPONENTIAL VALUES OF SINES AND COSINES OF ANGLES

We will usually denote the sine series by S and cosine series by C after a little modification if required so that the resulting expression takes the exponential form.

The exponential series may be reduced to one of the Algebraic series, Geometric series, Arithmetic series, Binomial series, Logarithmic series, Sine series or Cosine series etc. For revision purposes we write the following series.

(i) $\sin z = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots\cdots$

(ii) $\cos z = 1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} - \frac{z^6}{\lfloor 6} + \cdots\cdots$

(iii) $e^z = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + \cdots\cdots$

(iv) $\frac{1 - z^n}{1 - z} = 1 + z + z^2 + z^3 + \cdots\cdots + z^{n-1}$

(v) $(1 + z)^n = 1 + nz + \frac{n(n-1)}{\underline{|2}} z^2 + \cdots\cdots$

(vi) $\text{Log}(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} \quad \cdots\cdots$

**EXAMPLE 1** Sum up the infinite series $\cos 2\theta + \frac{1}{3}\cos 6\theta + \frac{1}{5}\cos 10\theta + \cdots \propto$.

**SOLUTION** Let

$$C = \cos 2\theta + \frac{1}{3}\cos 6\theta + \frac{1}{5}\cos 10\theta + \cdots\cdots$$

and $$S = \sin 2\theta + \frac{1}{3}\sin 6\theta + \frac{1}{5}\sin 10\theta + \cdots\cdots$$

$$C + iS = e^{i2\theta} + \frac{1}{3}e^{i6\theta} + \frac{1}{5}e^{i10\theta} + \cdots\cdots$$

Let $e^{i2\theta} = z$

Then

$$(C + iS) = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\cdots \propto.$$

Now consider

$$\text{Log}\left(\frac{1 + z}{1 - z}\right) = \text{Log}(1 + z) - \text{Log}(1 - z)$$

$$= \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \cdots\cdots\right)$$

$$- \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} \cdots\cdots\right)$$

$$= 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\cdots\right)$$

$$\therefore\ C+iS = z+\frac{z^3}{3}+\frac{z^5}{5}+\ldots\ldots = \frac{1}{2}\,\mathrm{Log}\left(\frac{1+z}{1-z}\right)$$

$$= \frac{1}{2}\,\mathrm{Log}\left(\frac{1+e^{2i\theta}}{1-e^{2i\theta}}\right)$$

$$= \frac{1}{2}\,\mathrm{Log}\left[\frac{(1+e^{2i\theta})(1-e^{-2i\theta})}{(1-e^{2i\theta})(1-e^{-2i\theta})}\right]$$

$$= \frac{1}{2}\,\mathrm{Log}\left[\frac{1+e^{2i\theta}-e^{-2i\theta}-1}{1-(e^{2i\theta}+e^{-2i\theta})+1}\right]$$

$$= \frac{1}{2}\,\mathrm{Log}\left[\frac{2\,i\sin 2\theta}{2-2\cos 2\theta}\right]$$

$$= \frac{1}{2}\,\mathrm{Log}\left[\frac{i\sin 2\theta}{2\sin^2\theta}\right]$$

$$= \frac{1}{2}\,\mathrm{Log}\,[i\cot\theta]$$

$$= \frac{1}{2}\,[\log\cot\theta+\mathrm{Log}\,i]$$

$$C+iS = \frac{1}{2}\,[\log\cot\theta+\log 1+i\,\pi/2]$$

$$= \frac{1}{2}\log\cot\theta+i\frac{\pi}{4}$$

$$\therefore\quad C = \frac{1}{2}\log\cot\theta,$$

and incidently,

$$S = \frac{\pi}{4}$$

**EXAMPLE 2** Find the sum upto infinity

$$1+\frac{1}{2}\cos\alpha+\frac{1\cdot 3}{2\cdot 4}\cos 2\alpha+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\alpha+\ldots\ldots\infty$$

**SOLUTION** Let

$$C = 1 + \frac{1}{2}\cos\alpha + \frac{1 \cdot 3}{2 \cdot 4}\cos 2\alpha + \cdots\cdots$$

and $$S = \frac{1}{2}\sin\alpha + \frac{1 \cdot 3}{2 \cdot 4}\sin 2\alpha + \cdots\cdots$$

so that

$$C + iS = 1 + \frac{1}{2}e^{i\alpha} + \frac{1 \cdot 3}{2 \cdot 4}e^{i2\alpha} + \frac{1 \cdot 3 \cdot 4}{2 \cdot 4 \cdot 6}e^{i3\alpha} = (1 - e^{i\alpha})^{-1/2}$$

Using Binomial series.

$$= (1 - \cos\alpha - i\sin\alpha)^{-1/2}$$

$$= [2\sin\alpha/2\,(\sin\alpha/2 - i\cos\alpha/2)]^{-1/2}.$$

$$= \frac{1}{\sqrt{2\sin\alpha/2}}\left[\cos\left(\frac{\alpha}{2} - \pi/2\right) + i\sin(\alpha/2 - \pi/2)\right]^{-\frac{1}{2}}$$

$$= \sqrt{\frac{1}{2\sin(\alpha/2)}}\left[\cos(\pi/4 - \alpha/4) + i\sin(\pi/4 - \alpha/4)\right]$$

Now equating real part.

$$C = \frac{\cos(\pi/4 - \alpha/4)}{\sqrt{2\sin(\alpha/2)}}$$

Incidently

$$S = \frac{\sin(\pi/4 - \alpha/4)}{\sqrt{2\sin(\alpha/2)}}$$

**EXAMPLE 3** Sum the series

$$\cos\alpha + x\cos(\alpha + \beta) + x^2\cos(\alpha + 2\beta) + x^3\cos(\alpha + 3\beta) + \cdots\cdots \text{ to } \infty.$$

where $|x e^{i\beta}| < 1$.

**SOLUTION**

Let $C = \cos\alpha + x\cos(\alpha + \beta) + \cdots\cdots$ to $\infty$.

and $S = \sin\alpha + x\sin(\alpha + \beta) + x^2\sin(\alpha + 2\beta) + \cdots\cdots$ to $\infty$.

So that

$$C + iS = e^{i\alpha} + xe^{i(\alpha+\beta)} + x^2e^{i(\alpha+2\beta)} + \ldots\ldots \infty.$$

$$C + iS = e^{i\alpha}[1 + (xe^{i\beta}) + (xe^{i\beta})^2 + (xe^{i\beta})^3 + \ldots\ldots \infty]$$

$$= \frac{e^{i\alpha}}{1 - xe^{i\beta}} \quad \text{as} \quad |xe^{i\beta}| < 1, \qquad \ldots\ldots (1)$$

by using the sum formula for infinite geometric series.

i.e. $S_\infty = \dfrac{a}{1 - r}$

Now in order to separate in real and imaginary parts we will multiply the R.H.S. by the conjugate of denominator to top and bottom of (1).

$$C + iS = \frac{e^{i\alpha}(1 - xe^{-i\beta})}{(1 - xe^{i\beta})(1 - xe^{-i\beta})}$$

$$= \frac{e^{i\alpha} - xe^{i(\alpha-\beta)}}{1 + x^2 - x(e^{i\beta} + e^{-i\beta})}$$

$$= \frac{[\cos\alpha - x\cos(\alpha-\beta)] + i[\sin\alpha - x\sin(\alpha-\beta)]}{1 + x^2 - 2x\cos\beta}$$

equating real part.

$$C = \frac{\cos\alpha - x\cos(\alpha-\beta)}{1 - 2x\cos\beta + x^2}$$

Incidently we have also calculated

$$S = \frac{\sin\alpha - x\sin(\alpha-\beta)}{1 - 2x\cos\beta + x^2}$$

**EXAMPLE 4** Sum the series

$$1 - \frac{x^2}{\underline{|2}}\cos 2\theta + \frac{x^4}{\underline{|4}}\cos 4\theta - \frac{x^6}{\underline{|6}}\cos 6\theta + \ldots\ldots \infty$$

**SOLUTION** Let

$$C = 1 - \frac{x^2}{\underline{|2}}\cos 2\theta + \frac{x^4}{\underline{|4}}\cos 4\theta - \frac{x^6}{\underline{|6}}\cos 6\theta + \ldots\ldots \infty.$$

$$\text{and } S = \frac{-x^2}{\underline{|2}} \sin 2\theta + \frac{x^4}{\underline{|4}} \sin 4\theta - \frac{x^6}{\underline{|6}} \sin 6\theta + \ldots\ldots \infty.$$

So that

$$C + iS = 1 - \frac{x^2}{\underline{|2}} e^{2i\theta} + \frac{x^4}{\underline{|4}} e^{4i\theta} - \frac{x^6}{\underline{|6}} e^{6i\theta} + \ldots\ldots \infty.$$

$$= 1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \frac{z^6}{\underline{|6}} + \ldots\ldots \infty.$$

where $z = xe^{i\theta}$

$$= \cos z = \cos (xe^{i\theta})$$

$$C + iS = \cos (xe^{i\theta})$$

$$= \cos (x \cos \theta + i x \sin \theta)$$

$$= \cos (x \cos \theta) \cos (i x \sin \theta)$$

$$- \sin (x \cos \theta) \sin (i x \sin \theta)$$

$$= \cos (x \cos \theta) \cosh (x \sin \theta) - i \sin (x \cos \theta) \sinh (x \cos \theta)$$

Equating real and Imaginary parts.

$$C = \cos (x \cos \theta) \cosh (x \sin \theta)$$

$$S = - \sin (x \cos \theta) \sinh (x \sin \theta)$$

**EXAMPLE 5** Sum the series

$$\sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta - \frac{1}{4} \sin 4\theta + \ldots\ldots \infty.$$

**SOLUTION** Let

$$S = \sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta - \frac{1}{4} \sin 4\theta + \ldots\ldots \infty.$$

$$\text{and } C = \cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \frac{1}{4} \cos 4\theta + \ldots\ldots \infty.$$

So that

$$C + iS = e^{i\theta} - \frac{1}{2} e^{i2\theta} + \frac{1}{3} e^{i3\theta} - \frac{1}{4} e^{i4\theta} + \ldots\ldots \infty.$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots \infty.$$

where $z = e^{i\theta}$.

$$= \text{Log}\,(1+z)$$

$$= \text{Log}\,(1+e^{i\theta}) = \text{Log}\,(1+\cos\theta + i\sin\theta)$$

$$= \text{Log}\,[\,2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2]$$

$$= \text{Log}\,(2\cos\theta/2) + \text{Log}\left[\cos\frac{\theta}{2} + i\sin\theta/2\right]$$

$$C + iS = \text{Log}\,(2\cos\theta/2) + \log\sqrt{(\cos^2\theta/2 + \sin^2\theta/2)} + i\tan^{-1}\left[\frac{\sin\theta/2}{\cos\theta/2}\right]$$

$$= \log 2 + \log\cos\theta/2 + \log 1 + i\,\theta/2.$$

$$= \log 2 + \log\cos\theta/2 + i\,\theta/2.$$

Equating real and Imaginary parts.

$$C = \log 2 + \log(\cos\theta/2)$$

$S = \theta/2$, the required sum of the series.

## EXERCISE 1.9

Prove that

1. $\cos x + \cos 2x + \cos 3x + \cdots\cdots$ upto n terms $= \dfrac{\cos\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\frac{x}{2}}$

2. $\dfrac{\sin x - \sin 2x + \sin 3x - \cdots \text{ to n terms}}{\cos x - \cos 2x + \cos 3x - \cdots \text{ to n terms}} = \tan\left\{\left(\frac{n+1}{2}\right)(\pi + x)\right\}$

3. Sum to n terms the series:-

   (i) $C = \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \cdots\cdots$

   (ii) $S = \cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cos^3\theta\sin 3\theta + \cdots\cdots$

**4.** Sum to n terms:-

(i) $\frac{\sin\theta}{\cos\theta} + \frac{\sin 2\theta}{\cos^2\theta} + \frac{\sin 3\theta}{\cos^3\theta} + \cdots\cdots$

(ii) $\frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \cdots\cdots$

**5.** Sum to $n+1$ terms the series:-

$$1 + n\cos x + \frac{n(n-1)}{\underline{|2}}\cos 2x + \frac{n(n-1)(n-2)}{\underline{|3}}\cos 3x + \cdots\cdots$$

**6.** Sum to $n+1$ terms the series:-

$1 + c\cos x + c^2\cos 2x + \cdots\cdots$

**7.** Sum to n terms the series:-

(i) $3\cos\theta + 5\cos 2\theta + 7\cos 3\theta + 9\cos 4\theta + \cdots\cdots$

(ii) $3\sin\theta + 5\sin 2\theta + 7\sin 3\theta + 9\sin 4\theta + \cdots\cdots$

Find the sum upto infinity of the following series:-

**8.** $\sin\theta\sin\theta + \sin^2\theta\sin 2\theta + \sin^3\theta\sin 3\theta + \cdots\cdots$

**9.** $\sin\theta\cos\theta + \sin^2\theta\cos 2\theta + \sin^3\theta\cos 3\theta + \cdots\cdots$

**10.** $1 + \frac{1}{2}\cos 2\theta - \frac{1}{2\cdot 4}\cos 4\theta + \frac{1\cdot 3}{2\cdot 4\cdot 6}\cos 6\theta - \cdots\cdots$

**11.** $\frac{1}{2}\sin 2\theta - \frac{1}{2\cdot 4}\sin 4\theta + \frac{1\cdot 3}{2\cdot 4\cdot 6}\sin 6\theta - \cdots\cdots$

**12.** $1 - \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\theta + \cdots\cdots$

**13.** $\frac{1}{2}\sin\theta - \frac{1\cdot 3}{2\cdot 4}\sin 2\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\theta - \cdots\cdots$

**14.** $\cos\theta - \frac{\cos(\theta+2\varphi)}{\underline{|3}} + \frac{\cos(\theta+4\varphi)}{\underline{|5}} - \cdots\cdots$

**15.** $1 + x\cos\alpha + \frac{x^2}{\underline{|2}}\cos 2\alpha + \frac{x^3}{\underline{|3}}\cos 3\alpha + \cdots\cdots$

16. $x\cos\alpha + \frac{x^2}{2}\cos 2\alpha + \frac{x^2}{3}\cos 3\alpha + \ldots\ldots$

17. $\sin x - \frac{1}{2}\sin 3x + \frac{1}{3}\sin 5x - \ldots\ldots$

18. $\cos\theta + \frac{\cos\theta}{1}\cos 2\theta + \frac{\cos^2\theta}{\underline{|2}}\cos 3\theta + \frac{\cos^3\theta}{\underline{|3}}\cos 4\theta + \ldots\ldots$

19. $\cos\theta + \frac{\sin\theta}{1}\cos 2\theta + \frac{\sin 2\theta}{\underline{|2}}\cos 3\theta + \ldots\ldots$

20. $x\sin\theta - \frac{x^2\sin 2\theta}{2} + \frac{x^3\sin 3\theta}{3} - \ldots\ldots$

21. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta + \ldots\ldots$

22. $\sin\theta - \frac{\sin 2\theta}{\underline{|2}} + \frac{\sin 3\theta}{\underline{|3}} + \ldots\ldots$

23. $1 - \frac{\cos 2\theta}{\underline{|2}} + \frac{\cos 4\theta}{\underline{|4}} - \ldots\ldots$

24. $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta + \ldots\ldots$

25. $c\cos\theta + \frac{c^3}{3}\cos 3\theta + \frac{c^5}{5}\cos 5\theta + \ldots\ldots$

# 2

# GROUP THEORY

## 2.1 INTRODUCTION

In the awake of 20th century, the introduction of the concept of a set has motivated the study of mathematics to its axiomatic gravity. This approach has not only solved some old problems but has opened new fields of independent study. New structures are formed. The role of mapping from one set to another is vital in the formation of new structures and their study. The structure of a group has stood the test of the time and thus has attained the status of being included in the first course on Abstract Algebra in the University Curricula.

### 2.1.1 MAPPING (FUNCTION)

Let X and Y be two non-empty sets. A rule $\phi$ or a technique of correspondence under which the elements of set X are connected or mapped into the elements of set Y is called a **mapping** from X **into** Y, if

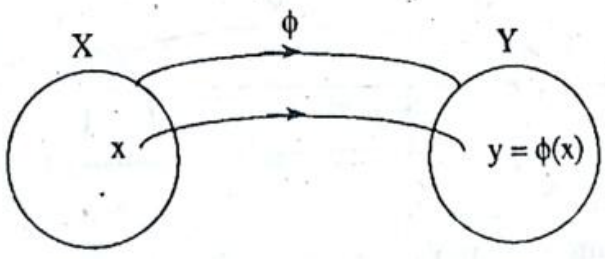
for each element $x \in X$, there exists a unique element $y = \phi(x) \in Y$.

Symbolically we write it as $\phi : X \to Y$.

Such a rule of correspondence may be **many-one** but not **one-many.**

$$\text{i.e., } x_1 = x_2 \Rightarrow \phi(x_1) = \phi(x_2),\ x_1, x_2 \in X.$$

The element $y = \phi(x)$ is called the **image** of x under the mapping $\phi$ and x is called the **pre-image** of y under the mapping $\phi$.




The mapping $\phi$ is called **over** (or on) X if Y = X.

The mapping $\phi$ is called **real valued** if Y = R, the set of real numbers and complex valued if Y = $\mathbb{C}$, the complex set.

### 2.1.2 KINDS OF MAPPINGS

Let $\phi$ be a mapping from X into Y.

i.e.. $\phi : X \longrightarrow Y$, then

(i) $\phi$ (X) is a subset of Y, called the **range** of $\phi$.

(ii) Whole of the set X forms the **domain** of $\phi$.

(iii) Whole of the set Y may or may not be the range of $\phi$.

(iv) $|\phi(X)| \leq |X|$, if x is finite.

(v) $\phi$ is understood as a subset of X × Y in which the element of X is not repeated.

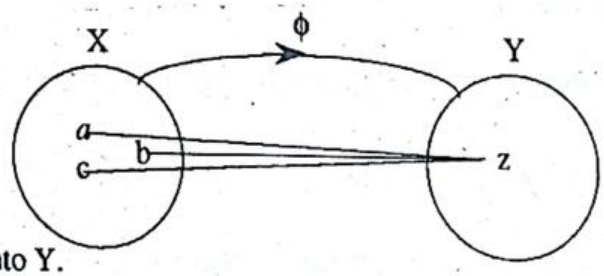
K1 : Rule $\phi$ is called a mapping from X onto Y, if $\phi(X) = Y$.

or

There is left no element of Y which does not have its pre-image in X.

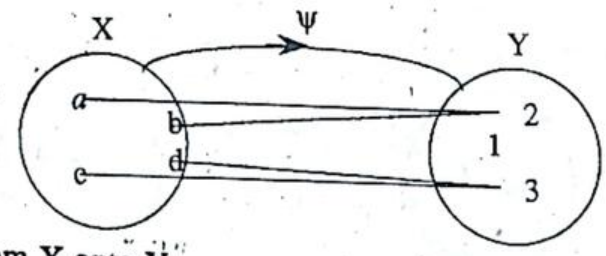
Such a map is also called **surjective.**

For example, if X = {a, b, c} and Y = {z}, then $\phi : X \to Y$ defined as,




is from X onto Y.

If X = {*a*, b, c, d} and Y = {1, 2, 3}, then $\varphi : X \to Y$ defined as,




is not from X onto Y.

Note that every mapping is from its domain onto its range.

$K_2$ : Rule $\phi$ is called one-to-one $(1-1)$ if distinct elements of X have their distinct images in Y.

or

one element of Y does not have its more than one pre–image

or

if the images of any two elements of X are not distinct then the elements themselves are not distinct

$$\text{i.e. } \forall x_1, x_2 \in X, \quad \phi(x_1) = \phi(x_2) \Rightarrow x_1 = x_2$$

Such a mapping is also called **injective** in which $|\phi(X)| \leq |Y|$, if X and Y are finite.

$K_3$ : Rule $\phi$ is called one-to-one correspondence if $\phi$ is $(1-1)$ and onto Y both. Such a map, is called **bijective** in which $|X| = |Y|$, if X is finite.

$K_4$ : "If Y is a singleton set then $\phi$ is **constant** and evidently $\phi$ is onto Y as well.

$K_5$ : $\phi$ is called an **identity map** over X $(Y = X)$, if, $\forall x \in X$, $\phi(x) = x$. It leaves every element of X unchanged and is denoted by $\phi_e$.

$\phi_e$ is clearly $(1-1)$ and onto over the set X.

### 2.1.3 DEFINITION

Let $\phi : X \to Y$ be a mapping from X into Y. A map $\psi : Y \to X$ is said to be inverse of $\phi$ if, $D_\phi = R_\psi$ and $R_\phi = D_\psi$. We denote $\psi = \phi^{-1}$.

**NOTE**

1. If $\phi$ is not $(1-1)$ then its inverse would be one-many. In this case, $\phi^{-1}$ does not qualify to be a mapping from Y into X.
2. If $\phi^{-1}$ is a map from Y into X, then it is onto X, because, $\phi^{-1}(Y) = X$.
3. If $\phi$ is a map onto Y and B is a subset of Y, then
$$\phi^{-1}(B) = \{x \in X : \phi(x) \in B\}$$
4. If $\phi \subseteq X \times Y$ then $\phi^{-1} \subseteq Y \times X = \{(y, x) : y \in Y, x \in X\}$.

### 2.1.4 THEOREM

Let $\psi : X \to Y$ be a mapping from X into Y. Then $\psi^{-1}$ is a mapping from Y into X if and only if $\psi$ is $(1-1)$ and onto Y both.

### 2.1.5 DEFINITION (Equality)

Let $\phi, \psi : X \to Y$ be two mappings from X into Y. Then $\phi = \psi$, if and only if,

(i) $D_\phi = D_\psi$ (ii) $R_\phi = R_\psi$ and (iii) $\forall x \in X, \phi(x) = \psi(x)$.

## EXAMPLES

1. Let $\phi : N \to N$ be a rule defined by $\phi(n) = 2n$, for every $n \in N$, the set of all natural numbers. Then for $n_1, n_2 \in N$,

$$n_1 = n_2 \Rightarrow 2n_1 = 2n_2 \Rightarrow \phi(n_1) = \phi(n_2)$$

and $\phi(n_1) = \phi(n_2) \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$ which proves that $\phi$ is a mapping which is $(1 - 1)$. $\phi$ is not onto N because odd natural numbers have no pre-images under $\phi$.

2. If $\mathbb{C} = \{x + iy : x,y \in R\}$ and $V_2 = \{(x,y) : x,y \in R\}$ be two sets of complex numbers and all the ordered pairs in the Euclidean plane respectively.
A maping $\psi : \mathbb{C} \longrightarrow V_2$ is defined by

$$\psi(x + iy) = (x,y), \forall\ x + iy \in \mathbb{C}$$

Then $\psi$ is a $(1 - 1)$ and onto $V_2$ and hence a $(1 - 1)$ correspondence.

3. If a map $\theta : \mathbb{C} \to M_2$ is defined by $\theta(x + iy) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $\forall\ x + iy \in \mathbb{C}$

where $M_2$ stands for the set of all $2 \times 2$ matrices over the real field R,

then $\theta$ is $(1 - 1)$ but not onto, because the matrices of $M_2$ which are not diagonal have no pre-images in $\mathbb{C}$ under $\theta$.

4. The map $\psi : R \longrightarrow R$ defined by

$$\psi(m) = 2^m, \qquad \forall m \in R,$$

is $(1 - 1)$ but not onto R. $\psi$ is onto $R^+$, the set of all positive real numbers.

5. The map $\psi : R \longrightarrow R$ defined by $\psi(m) = 2m + 1$, $\forall m \in R$, is $(1 - 1)$ because,

$$\psi(m_1) = \psi(m)$$
$$\Rightarrow 2m_1 + 1 = 2m + 1,$$
$$\Rightarrow m_1 = m, \qquad \forall m, m_1 \in R.$$

$\phi$ is onto R because, for every $\psi(m) = t = 2m+1$, there exists $m = \frac{t-1}{2}$, in R.

6. Let $\Omega = \{1,2,3\}$ be the set. The permutation $p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is a map over the set $\Omega$ defined by $p(1) = 2$, $p(2) = 3$, $p(3) = 1$.
p is therefore $(1 - 1)$ and onto over $\Omega$.

Similarly all the permutations over the elements of a set define mappings $(1 - 1)$ and onto over the set itself.

## SOME PROPERTIES OF MAPPINGS

### 2.1.6. THEOREM

Let $\psi : X \to Y$ be a mapping from X into Y. If A and B are two subsets of X, then

(i) $\psi(A \cap B) \subseteq \psi(A) \cap \psi(B)$

(ii) $\psi(A \cup B) = \psi(A) \cup \psi(B)$

and (iii) $\psi(A - B) \subseteq \psi(A) - \psi(B)$

**PROOF** *(i)* $\psi(A \cap B) = \{\psi(t) : t \in A \cap B\}$

$\Rightarrow \{\psi(t) : t \in A \text{ and } t \in B\}$

$\Rightarrow \{\psi(t) : \psi(t) \in \psi(A) \text{ and } \psi(t) \in \psi(B)\}$

$\Rightarrow \{\psi(t) : \psi(t) \in \psi(A) \cap \psi(B)\}$,

which proves that $\psi(A \cap B) \subseteq \psi(A) \cap \psi(B)$

Its converse does not hold. If $f : R \to R$ defined by $f(n) = n^2$.

If $A = \{0, 1\}$ and $B = \{0, -1\}$

then $f(A \cap B) = f(0) = 0 \neq f(A) \cap f(0) = \{0, 1\}$

*(ii)* $\psi(A \cup B) = \{\psi(t) : t \in A \cup B\}$

$= \{\psi(t) : t \in A \text{ or } t \in B\}$

$= \{\psi(t) : \psi(t) \in \psi(A) \text{ or } \psi(t) \in \psi(B)\}$

$= \{\psi(t) : \psi(t) \in \psi(A) \cup \psi(B)\}$

$\Rightarrow \psi(A \cup B) \subseteq \psi(A) \cup \psi(B)$

For the converse, if $t \in \psi(A) \cup \psi(B)$,

then $t \in \psi(A)$ or $t \in \psi(B)$.

$\Rightarrow \exists x$ in A or in B such that $\psi(x) = t$ or $t \in \psi(A \cup B)$

$\Rightarrow \psi(A) \cup \psi(B) \subseteq \psi(A \cup B)$

It proves the equality *(ii)*.

*(iii)* $\psi(A - B) = \{\psi(t) : t \in A - B\}$

$= \{\psi(t) : t \in A \text{ and } t \notin B\} = \{\psi(t) : t \in (A \cap B')\}$

$= \{\psi(t) : \psi(t) \in \psi(A) \text{ and } \psi(t) \notin \psi(B)\}$

$= \{\psi(t) : \psi(t) \in \psi(A) - \psi(B)\}$, by (i)

$\Rightarrow \psi(A - B) \subseteq \psi(A) - \psi(B)$

The fact that the converse does not hold is supported by the example shown in (i)

**REMARKS:**

1. For empty subset $\phi$ of X, $\psi(\phi)$ is the empty subset of Y.
2. If A, B are two subsets of X where $A \cap B = \phi$ and $\psi$ is constant then $\psi(A \cap B) \neq \psi(A) \cap \psi(B)$.
3. $\psi$ preserves the union of sets and not their intersection and difference.

### 2.1.7. THEOREM

Let $\psi : X \to Y$ be a mapping from X onto Y. If A and B are two subsets of Y, then

*(i)* $\psi^{-1}(A \cap B) = \psi^{-1}(A) \cap \psi^{-1}(B)$

(i) $\psi^{-1}(A \cup B) = \psi^{-1}(A) \cup \psi^{-1}(B)$
(ii) $\psi^{-1}(A - B) = \psi^{-1}(A) - \psi^{-1}(B)$

**PROOF**

*(i)* $\psi^{-1}(A \cap B) = \{x \in X : \psi(x) \in A \cap B\}$
$\Leftrightarrow \{x \in X : \psi(x) \in A \text{ and } \psi(x) \in B\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) \text{ and } x \in \psi^{-1}(B)\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) \cap \psi^{-1}(B)\}$
Hence $\psi^{-1}(A \cap B) = \psi^{-1}(A) \cap \psi^{-1}(B)$

*(ii)* $\psi^{-1}(A \cup B) = \{x \in X : \psi(x) \in A \cup B\}$
$\Leftrightarrow \{x \in X : \psi(x) \in A \text{ or } \psi(x) \in B\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) \text{ or } x \in \psi^{-1}(B)\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) \cup \psi^{-1}(B)\}$
Hence $\psi^{-1}(A \cup B) = \psi^{-1}(A) \cup \psi^{-1}(B)$

*(iii)* $\psi^{-1}(A - B) = \{x \in X : \psi(x) \in A - B\}$
$\Leftrightarrow \{x \in X : \psi(x) \in A \text{ and } \psi(x) \notin B\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) \text{ and } x \notin \psi^{-1}(B)\}$
$\Leftrightarrow \{x \in X : x \in \psi^{-1}(A) - \psi^{-1}(B)\}$
Hence $\psi^{-1}(A - B) = \psi^{-1}(A) - \psi^{-1}(B)$

## 2.1.8. COMPOSITION OF MAPPINGS

Let $\theta : X \to Y$ be a map from X into Y and $\psi : Y \to Z$ from Y into Z, then the rule $\psi \circ \theta : X \to Z$, from X into Z defined by

$$\psi \circ \theta(x) = \psi(\theta(x)), \quad \forall x \in D_\theta$$

is said to be a product of $\theta$ with $\psi$. This product rule is known as the **composition** of $\theta$ with $\psi$.

## 2.1.9. THEOREM

Let $\theta : X \to Y$ and $\psi : Y \to Z$ be mappings from X into Y and Y into Z respectively. Then their product $\psi \circ \theta$ is a mapping from X into Z.

**PROOF.** Take $x_1, x_2 \in X$. Then $\theta(x_1), \theta(x_2) \in Y$ and $\psi(\theta(x_1)), \psi(\theta(x_2)) \in Z$.

$x_1 = x_2 \Rightarrow \theta(x_1) = \theta(x_2)$ $\quad \because \theta$ is a map
$\Rightarrow \psi(\theta(x_1)) = \psi(\theta(x_2))$ $\quad \because \psi$ is a map.
$\Rightarrow \psi \circ \theta(x_1) = \psi \circ \theta(x_2)$ $\quad \forall x_1, x_2 \in X$
$\Rightarrow \psi \circ \theta$ is a map from X into Z.

It may be noted that $D_{\psi \circ \theta} = D_\theta$ and $R_{\psi \circ \theta} \subseteq R_\psi$.

COR. If $\theta$ and $\psi$ are mappings over a set X then their products $\psi \circ \theta$ and $\theta \circ \psi$ are also mappings over X.

### 2.1.10 THEOREM

If $\phi$ is a mapping over a set X and $\phi^{-1}$ is its inverse map, then

$$\phi \circ \phi^{-1} = \phi_e = \phi^{-1} \circ \phi$$

PROOF We know by definition that, if $\phi(t) = x$ then $\phi^{-1}(x) = t$.

Also, $\phi^{-1} \circ \phi(t) = \phi^{-1}(\phi(t)) = \phi^{-1}(x)$

$= t, \quad \forall t \in X.$

and $\phi \circ \phi^{-1}(x) = \phi(\phi^{-1}(x)) = \phi(t) = x, \quad \forall x \in X$

where $\phi_e(x) = x, \quad \forall x \in X.$

Hence $\phi \circ \phi^{-1} = \phi_e = \phi^{-1} \circ \phi$, for any $\phi \in M(X)$,

where M(X) stands for the set of all mappings over X.

### 2.1.11 THEOREM

Let X be a non-empty set and M(X) be the set of all mappings over X.

If $\psi_1, \psi_2 \in M(X)$, then

(i) $\psi_1 \circ \psi_2 (\psi_2 \circ \psi_1)$ is $(1-1)$ if, $\psi_1$ and $\psi_2$ both are $(1-1)$.

(ii) $\psi_1 \circ \psi_2 (\psi_2 \circ \psi_1)$ is onto if $\psi_1$ and $\psi_2$ both are onto X.

PROOF (i) Suppose $\psi_1$ and $\psi_2$ are mappings $(1-1)$ over X.

If $x_1, x_2 \in X$, then

$\psi_1(x_1) = \psi_1(x_2) \Rightarrow x_1 = x_2$. And $\psi_2(x_1) = \psi_2(x_2) \Rightarrow x_1 = x_2$

Take $\psi_1 \circ \psi_2(x_1) = \psi_1 \circ \psi_2(x_2)$

$\Rightarrow \psi_1(\psi_2(x_1)) = \psi_1(\psi_2(x_2))$.

$\Rightarrow \psi_2(x_1) = \psi_2(x_2), \quad \because \psi_1$ is $(1-1)$

$\Rightarrow x_1 = x_2, \quad \because \psi_2$ is $(1-1)$

Hence $\psi_1 \circ \psi_2$ is also $(1-1)$. Similarly $\psi_2 \circ \psi_1$ is $(1-1)$.

(ii) Suppose that $\psi_1, \psi_2 \in M(X)$ are onto.

Since $\psi_1$ is onto X, therefore $\psi_1(X) = X$.

Since $\psi_2$ is onto X, therefore $\psi_2(X) = X$.

Then

$$\psi_1 \circ \psi_2(X) = \psi_1(\psi_2(X))$$

$= \psi_1(X)$ , as $\psi_2$ is onto

$= X$ , as $\psi_1$ is onto

which implies that $\psi_1 \circ \psi_2$ is onto X. Similarly it can be shown that $\psi_2 \circ \psi_1$ is also onto X.

**EXAMPLE** Let f and g be two mappings over the set R of all real numbers defined by

$f(m) = 2m - 1$ and $g(m) = m^2 + 7m + 2$ respectively.

Then define the rule for mappings,

*(i)* $f \circ g$ *(ii)* $g \circ f$ *(iii)* $f^{-1}$ *(iv)* $g \circ f^2$

For *(i)*, by definition, $f \circ g(m) = f(g(m)), \ \forall \ m \in R$

$= f(m^2 + 7m + 2)$

$= 2(m^2 + 7m + 2) - 1$

$= 2m^2 + 14m + 4 - 1$

$= 2m^2 + 14m + 3$

Hence $f \circ g(m) = 2m^2 + 14m + 3$ is the required rule.

*(ii)* $g \circ f(m) = g(f(m)) = g(2m - 1)$

$= (2m - 1)^2 + 7(2m - 1) + 2$

$= 4m^2 + 1 - 4m + 14m - 7 + 2$

$= 4m^2 + 1 - 4m + 14m - 7 + 2$

$= 4m^2 + 10m - 4$

Thus $g \circ f(m) = 4m^2 + 10m - 4$, is the rule for (ii).

Note that $g \circ f \neq f \circ g$

(iii) $(m, 2m - 1) \in f, \ \forall \ m \in D_f$ and $f(m) = 2m - 1 \in R_f$.

Thus $(m + 1, 2m) \in f, \ \forall \ m \in D_f$.

$$\Rightarrow \left(\frac{m + 1}{2}, m\right) \in f$$

Thus, $f^{-1}(m) = \frac{m + 1}{2}$ is the rule of definition of $f^{-1}$ over R.

Note that

*(i)* $f^{-1}$ is a mapping over R. *(ii)* $f^{-1}$ is (1 – 1) and onto.

(iii) $f(f^{-1}(m)) = f\left(\frac{m + 1}{2}\right) = 2\left(\frac{m + 1}{2}\right) - 1 = m$

$$f^{-1} \circ f(m) = f^{-1}(f(m)) = f^{-1}(2m - 1) = \frac{(2m - 1) + 1}{2} = m$$

Hence $f \circ f^{-1} = f^{-1} \circ f =$ Identity map over R.

Since f is (1 – 1) and onto, therefore $f^{-1}$ is a mapping defined as above.

*(iv)* $g \circ f^2(m) = g \circ (f \circ f)(m) = g \circ f(f(m)) = g(f(2m-1)) = g(2(2m-1) - 1)$

$= g(4m - 2 - 1) = g(4m - 3) = 16m^2 - 24m + 9 + 28m - 21 + 2$

$= 16m^2 + 4m - 10$

That is, $g \circ f^2(m) = 2(8m^2 + 2m - 5)$

## EXERCISE 2.1

1. Let $\mathbb{C}$ be the set of all complex numbers and R, the set of all real numbers. A rule $f : \mathbb{C} \to R$ from $\mathbb{C}$ into R is defined by $f(z) = |z|$, $\forall\ z \in \mathbb{C}$. Then prove that
   *(i)* f is a mapping from $\mathbb{C}$ into R.
   *(ii)* f is not (1 – 1).
   *(iii)* f is not onto R.

2. If A = {x,y,z}, then write down all the mappings which are (1 – 1) and onto A itself.

3. If B = {1,2,3} and C = {*a*, b}, then find all the constant mappings from B into C.

4. Let S and T be two non-empty sets. A rule $\phi : S \times T \longrightarrow T \times S$ from S x T into T x S is defined by $\phi(s,t) = (t,s)$, $\forall\ s \in S, t \in T$. Prove that $\phi$ is a (1 – 1) correspondence.

5. For a fixed pair $a, b \in R$, $a \neq 0$, a mapping $f : R \longrightarrow R$, over R is defined by $f(m) = am + b$. Prove that f is (1 – 1) and onto.

6. Let $f : R \longrightarrow R$ be a mapping over the set R of all real numbers, defined by $f(x) = x^3 - x$. Show whether f is (i) an onto mapping (ii) a (1 – 1) mapping?

7. Show, whether a mapping $\alpha : R \longrightarrow R$, over R defined by $\alpha(x) = 2x - 1$, $\forall x \in R$, is a (1 – 1) correspondence?

8. If $f, g : R^* \to R^*$ be mappings over $R^* = R - \{0\}$ defined by $f(x) = x^2 - 3$ and $g(x) = 3x - 1$. Then find *(i)* $f \circ g(x)$ *(ii)* $g \circ f(x)$ *(iii)* $g \circ f(2)$ *(iv)* $f \circ g(0)$ *(v)* $g^{-1}(x)$ *(vi)* $f^{-1}(m)$ and show whether $f^{-1}$ is a mapping?

9. If f and g are mappings over R, defined by $f(x) = x^2$ and $g(x) = x - 1$, then find
   (i) $f \circ g(x)$ (ii) $g \circ (f \circ g)(x)$ (iii) $f \circ (g \circ f)(x)$ (iv) $g \circ g(x)$ and (v) $f \circ f(x)$

10. Prove, whether any mapping from a set A onto B is essentially one–to–one?

11. Verify, whether $f : R \to R$, defined by $f(x) = 3x - 5$ is one-to-one and also onto (bijective) mapping.

12. Prove that a mapping $f : R^* \longrightarrow R^*$, defined by $f(x) = \frac{1}{x}$ is (1 – 1) and onto.

13. If $f : X \longrightarrow Y$ is a mapping (1 – 1) and onto Y. Test, whether $f^2$, $f^3$ are (1 – 1) and onto or not?

**14.** If $f : X \longrightarrow Y$ is a mapping, then prove that

(i) $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B)$ when $A \subseteq Y, B \subseteq Y$.

(ii) $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$, $A \subseteq Y, B \subseteq Y$.

**15.** Let $f : X \longrightarrow Y$ be a mapping. If $A \subset X$ and $B \subset Y$, then show that

(i) $f(f^{-1}(B) \subseteq B$

(ii) if f is onto then $f(f^{-1}(B)) = B$

(iii) if f is $(1-1)$ then $f^{-1}(f(A)) = A$,

**16.** Let $\Omega = \{1,2,3\}$. Then construct

(i) a mapping over $\Omega$.

(ii) a onto map over $\Omega$.

(iii) a $(1-1)$ correspondence over $\Omega$.

(iv) All the mappings over $\Omega$.

**17.** If $A = \{1, 2, 3\}$, then show that a onto map over A is essentially $(1-1)$.

**18.** Let W and N be the sets of all whole and natural numbers respectively.

A map $f^* : W \longrightarrow N$, is defined by

$$f(m) = \begin{cases} \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_n p_n \text{ , if } m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n} \\ 1 \text{ , if } m = 0 \end{cases}$$

Then (i) Find, $f(12)$, $f(36)$, $f(97)$, $f(100)$.

(ii) Is f one-to-one?

*(iii)* Is $f(m_1 \cdot m_2) = f(m_1) \cdot f(m_2)$ or not, $\forall\ m_1, m_2 \in W$

**19.** If g, f be real valued mappings with their domain R, the set of all real numbers, defined by $f(x) = x^2 + 1$ and $g(x) = x - 1$

and (i) $f \cdot g(x) = f(x) \cdot g(x)$.

(ii) $(f + g)(x) = f(x) + g(x)$

then (i) Find $f \cdot g(20)$

(ii) Find x if $f \cdot g(x) = 1$

(iii) Are $f \cdot g$ and $f + g$ injective?

(iv) Are $g \cdot f$ and $f + g$ surjective?

(v) Are $f \cdot g$ and $f + g$ bijective?

Note that mappings $f \cdot g$ and $f + g$ defined in *(i)* and *(ii)* are called pointwise multiplication and pointwise addition respectively over R.

* This mapping is known in the literature as Chawla's function after the name of a famous Pakistani mathematician Prof. L.M. Chawla.

## 2.2 BINARY OPERATION OVER A SET

Binary operation over a given non-empty set X is a mapping, inparticular whose domain is the set $X \times X$.

### 2.2.1 DEFINITION

Let X be a non-empty set. A mapping $* : X \times X \to Y$ from $X \times X$ into Y is said to be a binary operation defined on or over X, such that

$$* (x_1, x_2) = x_1 * x_2 \in Y, \text{ for all } x_1, x_2 \in X.$$

### 2.2.2 PROPERTIES OF BINARY OPERATIONS

### $P_1$ : CLOSURE

Let $* : X \times X \to Y$ be a binary operation over X. Then $*$ is said to be closed over X if $Y \subseteq X$. i.e., $\forall\ x_1, x_2 \in X$,

$$\forall\ x_1, x_2 \in X, \quad * (x_1, x_2) = x_1 * x_2 \in X.$$

or

'*' is a closed binary operation over X if

$* : X \times X \to X$ is a mapping from $X \times X$ into X.

**FOR EXAMPLE**

1. Addition (+) is a closed binary operation over the sets N, Q, R and $\mathbb{C}$.
2. Subtraction (–) is not closed over N, the set of all natural numbers.
3. Division (÷) is not closed over Q. the rationals.

### $P_2$ : ASSOCIATIVE

A binary operation '*' over the set X is said to be associative if;

$$\forall\ x_1, x_2, x_3 \in X \qquad (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$$

**FOR EXAMPLE**

The binary operations of + and × are associative over N,Z,Q,R, and $\mathbb{C}$, while '–' and ÷ are not associative.

## $P_3$ : COMMUTATIVE

Let a binary operation 'o' be defined over a non-empty set X. 'o' is said to be commutative over X, if

$$\forall\ x_1, x_2, \in X, \quad x_1 \circ x_2 = x_2 \circ x_1$$

**FOR EXAMPLE**

1. Ordinary addition (+) and multiplication (·) defined over the sets of real, complex, integers, rational and natural numbers are commutative.

2. The binary operation of addition over the set $M_m$ of all $m \times m$ square matrices is commutative but that of matrix multiplication is not commutative.

3. Composition of non-identity mappings or permutations over a finite set $\Omega$ is not commutative.

## $P_4$ : IDENTITY

Let X be a non-empty set and 'o' be a binary operation defined over X. The binary operation 'o' is said to observe the property of identity over X, if X contains an element e such that,

$$e \circ x = x = x \circ e, \quad \forall\ x \in X.$$

**FOR EXAMPLE**

1. Additive identity element of Z, Q, R, and $\mathbb{C}$, is 0 but 1 is the multiplicative identity over N, $Q^*$, $R^*$ and $\mathbb{C}^*$.

2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity over the set $\overline{M_2}$ of all $2 \times 2$ square non-singular matrices over the real field.

3. $0\underline{i} + 0\underline{j}$, the zero-vector, is the additive identity element of all vectors in the plane, because

$$\begin{aligned} (x\underline{i} + y\underline{j}) + (0\underline{i} + 0\underline{j}) &= (x + 0)\,\underline{i} + (y + 0)\underline{j} \\ &= x\underline{i} + y\underline{j} \\ &= (0\underline{i} + 0\underline{j}) + (x\underline{i} + y\underline{j}) \qquad \forall\ x, y \in R. \end{aligned}$$

## $P_5$ : INVERSE

A binary operation '*' over a non-empty set X is said to observe the inverse property if,

(i) '$*$' observes the identity property over X

and (ii) for every $x \in X$, there exists an element $x^{-1} \in X$ such that

$$x * x^{-1} = e = x^{-1} * x$$

x and $x^{-1}$ are called inverses of each other under the binary operation $*$.

**FOR EXAMPLE**

1. If $X = R$ and $x = 0 \in R$, then 0 does not have its inverse under the operation of multiplication of real numbers. Thus multiplication does not observe inverse property over R. If $X = R^*$, then the multiplication does observe the inverse property over $R^*$. Similar is the case for $Q^*$ and $\mathbb{C}^*$.

2. If $X = M_2$, the set of all $2 \times 2$ matrices over the real field the multiplication of matrices does not observe the inverse property over $M_2$ but the addition of matrices does observe the inverse property over $M_2$.

3. The binary operation of composition of functions over the set F of all functions over a set does not observe the inverse property. If $F^*$ is the set of all $(1 - 1)$ and onto functions then their composition observes the inverse property.

## $P_6$ : CANCELLATIVE

The binary operation '$*$' is said to be cancellative over a non-empty set X if,

(i) $*$ is right cancellative over X.

i.e., $x_1 * t = x_2 * t \Rightarrow x_1 = x_2$ $\quad x_1, x_2, t \in X.$

and (ii) $*$ is left cancellative over X.

i.e., $t * x_1 = t * x_2 \Rightarrow x_1 = x_2,$ $\quad$ for every $x_1, x_2, t \in X.$

Note that a binary operation is cancellative if it is both left and right cancellative.

**FOR EXAMPLE**

1. Addition is cancellative over the set R of all reals.

i.e. $m + 2 = n + 2 \Rightarrow m = n.$

$2 + m = 2 + n \Rightarrow m = n.$

Similarly multiplication over $R^*$ is cancellative but not over R

$a \cdot 0 = b \cdot 0 \Rightarrow a = b.$ $a, b \in R$

2. Matrix multiplication is not cancellative over the set of all 2 x 2 matrices but it is cancellative over the non-singular matrices $\overline{M}_2$. The matrix addition is cancellative over the set of all matrices.

### 2.2.3 THEOREM

Let e be the identity element of a non-empty set X under the binary operation $*$. Then e is a unique such element.

**PROOF** Suppose e is not unique. Then there exists $f \neq e$ in X such that

$$x * f = x = f * x, \forall\ x \in X.$$

But $x * c = x * x, \forall\ x \in X$, because e is the identity element of X. Thus

$$e * f = \begin{cases} e, \text{ if } f \text{ is identity} \\ f, \text{ if e is identity} \end{cases}$$

But $e * f$ is unique, therefore $f = e$. It provides contradiction to our supposition that $f \neq e$. Hence e is unique in X. Similarly, the product $f * e$ can either be considered.

### 2.2.4 THEOREM

Let $x^{-1}$ be inverse of x in X under the binary operation $*$. Then $x^{-1}$ is a unique such element in X, if '$*$' is cancellative over X.

**PROOF** Supposre '$*$' is cancellative over X and $x^{-1}$ be an inverse of x in X which is not unique. Then suppose there exists b, say $(\neq x^{-1})$ in X such that

$$b * x = e = x * b.$$

But
$$x^{-1} * x = e = x * x^{-1}.$$

$$\Rightarrow b * x = x^{-1} * x$$

$$\Rightarrow b = x^{-1} \text{ (by cancellative law)}$$

It provides contradiction. Hence $x^{-1}$ is unique inverse of x, for each $x \in X$.

Note that if $*$ is associative over X and contains the identity then uniqueness of the inverse is still valid.

## EXERCISE 2.2

1. Answer the following statements true or false. Support your answer by the proof.

   (i) On $Z^+$, $*$ defined by $a * b = c$, where c is atleast 5 more than $a + b$, is closed.

   (ii) On Z, $*$ defined by $a * b = a - b$, is a closed binary operation.

   (iii) On $Z^+$, $*$ defined by $a * b = 2ab$, is a commutative binary operation.

   (iv) On R, $*$ defined by $x * y = x + y - x \cdot y$ is a commutative and cancellative binary operation.

   (v) On R, $*$ defined by $a * b = 2a + 3b$, is not associative $(a, b \neq 0)$.

   (vi) On $Z^+$, o defined by $a \text{ o } b = a + b + a \cdot b$, is cancellative and associative.

   (vii) On $Z^-$, $*$ defined by $a * b = a \cdot b$, is a closed binary operation.

   (viii) Let P(X) be the power set of non-empty set X. On P(X), $*$ is defined by $A * B = A \cup B$, is an associative binary operation. $*$ is also commutative and observes the inverse property.

   (ix) On the power set P(X) of a non-empty set X, $*$ defined as $A * B = A \cap B$ is cancellative, prove it. It also observes the identity property.

2. Let $*$ be a binary operation which observes the inverse property over a non-empty set X. Then show that $(x^{-1})^{-1} = x$, for every $x \in X$

3. If '$\cdot$' is closed and associative over a non-empty set X and observes also the inverse and identity properties over X, then prove that

   (i) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, $\forall\ x, y \in X$.

   (ii) $(x.y)^{-1} = x^{-1}.y^{-1}$, if '$\cdot$' is commutative over X.

4. If $x \cdot y = y \cdot x$, for all $x, y \in X$, for a binary operation '$\cdot$' closed over X, which observes the identity and associative properties then prove that

   (i) $x \cdot y^{-1} = y^{-1} \cdot x$ (ii) $x^{-1} \cdot y = y \cdot x^{-1}$

   (iii) $x^2y = yx^2$ (iv) $xy^2 = y^2x$ (v) $x^n \cdot y^n = y^n \cdot x^n$

## 2.3. ALGEBRAIC STRUCTURES

The main objective here is to introduce the concept of a group. We gather together, the elementary knowledge which relates to the idea of a group and its development. For this purpose, we introduce first briefly the algebraic structures which are ancestors of a group.

### 2.3.1. ALGEBRAIC SYSTEM

The pair (G, $*$) of a non-empty set G together with the binary operation $*$, is said to form an algebraic system.

### 2.3.2. GROUPOID

The algebraic system (G, $*$) is said to be a groupoid if the binary operation $*$ has the closure property over the set G.

$$\text{i.e.,} \quad \forall \quad x, y \in G, \quad x * y \in G$$

### 2.3.3. SEMI-GROUP

The algebraic system (G, $*$) is a semi-group if,

(i) The binary operation * is closed over G, that is, (G, *) is a groupoid.

and (ii) '*' is associative over the set G.

Independently, the system (G, $*$) is called a semi-group if it observes the following ;

(i) $x * y \in G$ (closure)

(ii) $(x * y) * z = x * (y * z)$ (associative)

for all x, y, z in G.

### 2.3.4. MONOID

A semigroup (G, *) is called a Monoid, if it contains the *-identity of G. For example the set N of all natural numbers under multiplication is a Monoid.

### 2.3.5. GROUP

The system of group will be the centre of our attention in this chapter from now onwards.

#### 2.3.5.1. DEFINITION

Let (G, $*$) be an algebraic system. Then (G, $*$) is called a group if;

$G_1$ : '$*$' is closed over G.

i.e., for every $g_1, g_2$ in G, $g_1 * g_2 \in G$.

$G_2$ : '$*$' is associative over G.

i.e., for every $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$

$G_3$ : '$*$' observes the identity property over G.

That is, there exists an element $e \in G$ such that

$$e * g = g = g * e, \qquad \forall\ g \in G.$$

The element e is called, the identity element or some times, the neutral element of G.

$G_4$ : '$*$' observes the inverse property over G.

That is, for each element $g \in G$, there exists an element $g^{-1}$, the inverse of g, in G such that

$$g * g^{-1} = e = g^{-1} * g$$

From the definition, it is evident that a system of a groupoid with properties $G_2$, $G_3$ and $G_4$ forms a group. A semi-group with additional properties $G_3$ and $G_4$ forms the structure of a group. A monoid with property $G_4$ forms a group structure.

**It is important to note that;**

(i) . There is no concept of an empty group.

(ii) A group contains atleast the identity element.

(iii) The uniqueness of the identity element in a group follows by theorem 2.2.3.

(iv) The inverse of the identity element e is e itself in G.

(v) $(g^{-1})^{-1} = g$, for every g in group G.

**Proof.** Since, $g * g^{-1} = e = g^{-1} * g, \ \forall\ g \in G,$

therefore, by definition

g is the inverse of $g^{-1} \Rightarrow g = (g^{-1})^{-1}$

(vi) The cancellation laws (left as well as right) hold in a group.

(vii) The inverse of each element is unique, by theorem 2.2.4.

(viii) A group (G, $*$) is finite or infinite according as G, as a set, is finite or infinite respectively.

## 2.3.6. ABELIAN GROUP

A group (G, $*$) is said to be an Abelian group if '$*$' observes the commutative property over G.

that is, $g_1 * g_2 = g_2 * g_1$, for every $g_1, g_2 \in G$.

Such groups are commonly known as commutative groups. The name 'Abelian' is given after the name of a Norwegian mathematician N. H. Abel (1802 – 1829), as a recognition of his work in this field.

We agree on the following, for the sake of convenience, that,

(i) The binary operation of a group shall be understood to be that of multiplication and will be denoted by '.' unless given otherwise.

(ii) Instead of using (G, $*$), we shall be writing G alone.

(iii) The binary operation of an abstract abelian group will be taken '+'.

## 2.3.7 EXAMPLES

1. The set N of all natural numbers is a semi–group under addition (+) as well as under multiplication (·) of numbers.

2. The set Z of all integers is an abelian group under addition (+) and a semi-group under multiplication.

3. The set Q of all rational numbers, under the binary operation of addition (+), is an abelian group and a semi-group under multiplication (·). If $Q^* = Q - \{0\}$, then $Q^*$ is a group structure under multiplication.

4. The set R of all real numbers is an abelian group under the binary operation of addition (+) and a semi-group under multiplication (·). $R^*$, a subset of non–zero R, forms an abelian group under multiplication.

5. The set $\mathbb{C}$ of all complex numbers forms an abelian group under the binary operation of addition of complex numbers and a semi-group under the multiplication of complex numbers. $\mathbb{C}^* = \mathbb{C} - \{0\}$ forms an abelian group under multiplication.

**NOTE** The students are already familiar with the binary operations of addition (+) and multliplication (·) over the sets of numbers, they are advised verification of the bove mentioned respective structures. The groups from 2 – 5 are all infinite.

6. Let $G = \{1, -1, i, -i\}$. Then, under the binary operation of usual multiplication (·) it forms a group. Closure property is seen from the following multiplication table.

| · | 1 | −1 | i | −i |
|---|---|---|---|---|
| 1 | 1 | −1 | i | −i |
| −1 | −1 | 1 | −i | i |
| i | i | −i | −1 | 1 |
| −i | −i | i | 1 | −1 |

The student is advised to verify the following:

(i) The associative property of '·' over G.

(ii) 1 is the identity of the system G.

(iii) $-1$ and 1 are the inverses of their own

(iv) $-i$ is the inverse of i and i is the inverse of $-i$.

(v) G is an abelian group.

7. Let $U_3 = \{1, w, w^2\}$, where w is a complex cube root of unity. Then, under the binary operation of multiplication, $U_3$ forms an abelian group. The multiplication table given below defines the closure property,

| · | 1 | w | $w^2$ |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | 1 |
| $w^2$ | $w^2$ | 1 | w |

, $w^3 = 1$

Associative property is left to be verified by the reader.

1, is the identity in $U_3$, w is the inverse of $w^2$ and $w^2$ is the inverse of w. Abelian property can easily be verified, which is obvious from the symmetric nature of the table.

Similarly, the set $U_n$ of n, nth roots of unity forms an abelian group under the binary operation of multiplication (·).

$$U_n = \left\{ e^{\frac{2\pi mi}{n}} : m = 0, 1, 2, \ldots, n-1 \right\}$$

If $0 \le m_1, m_2 < n$ then

$$e^{\frac{2\pi m_1 i}{n}} \cdot e^{\frac{2\pi m_2 i}{n}} = \begin{cases} e^{\frac{2\pi(m_1 + m_2)i}{n}}, & \text{if } m_1 + m_2 < n. \\ e^{\frac{2\pi(m_1 + m_2 - n)i}{n}}, & \text{if } m_1 + m_2 \ge n. \end{cases}$$

For $m = 0$, $e^{2\pi oi} = \cos 0^0 + i \sin 0^0 = 1 + 0 = 1$, is the multiplicative identity of $U_n$.

$e^{\frac{2\pi(n - m_1)i}{n}}$ is the multiplicative inverse of $e^{\frac{2\pi m_1 i}{n}}$, and the vice versa because,

$$e^{\frac{2\pi(n-m_1)i}{n}} \cdot e^{\frac{2\pi m_1 i}{n}} = e^{\frac{2\pi (n - m_1 + m_1)i}{n}}$$

$$= e^{\frac{2\pi ni}{n}}$$

$$= e^{2\pi i} = \cos 2\pi + i \sin 2\pi.$$

$$= 1$$

Commutative property follows from the multiplication,

$$e^{\frac{2\pi m_1 i}{n}} e^{\frac{2\pi m_2 i}{n}} = e^{\frac{2\pi (m_1 + m_2)i}{n}} = e^{\frac{2\pi (m_2 + m_1)i}{n}}$$

$$= e^{\frac{2\pi m_2 i}{n}} \cdot e^{\frac{2\pi m_1 i}{n}}$$

Associative property is left as an exercise.

Hence $U_n$ forms a finite abelian group, for every positive integer n.

8. Let $\overline{M}_2 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in R \right\}$ be the set of all upper triangular $2 \times 2$ non-singular matrices. Then $\overline{M}_2$ forms a non-abelian group under the binary operation of matrix multiplication

defined by, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} \in \overline{M}_2.$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element in $\overline{M}_2$, because,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} \frac{c}{ac} & -\frac{b}{ac} \\ 0 & \frac{a}{ac} \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, because,

$$\begin{bmatrix} \frac{c}{ac} & -\frac{b}{ac} \\ 0 & \frac{a}{ac} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} \frac{c}{ac} & -\frac{b}{ac} \\ 0 & \frac{a}{ac} \end{bmatrix}$$

The associative property follows trivially and can be verified by the reader.

Thus $\overline{M}_2$ is a group under the binary operation of matrix multiplication. For commutativity,

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a'a & a'b + b'c \\ 0 & cc' \end{bmatrix}$$

Since $ab' + bc'$ is not equal to $a'b + b'c$ in general, therefore $\overline{M}_2$ is not an abelian group.

9. The set $V_2 = \{ x\underline{i} + y\underline{j} : x, y \in R \}$ of all the vectors in the Enclidean plane forms an abelian group under the binary operation of vector addition defined by

$$(x\underline{i} + y\underline{j}) + (x'\underline{i} + y'\underline{j}) = (x + x')\underline{i} + (y + y')\underline{j} \in V_2.$$

$0\underline{i} + 0\underline{j}$ is the Identity element of $V_2$

$(-x)\underline{i} + (-y)\underline{j}$ is the additive inverse of $x\underline{i} + y\underline{j}$

Associative and commutative properties are left for the reader to verify.

## 10. PERMUTATION GROUPS

### DEFINITION

Let $\Omega = \{1, 2, 3, \ldots, n\}$ be a finite set of n elements. Then a $(1 - 1)$ and onto mapping over $\Omega$ is called a permutation of $\Omega$.

Let $\Omega = \{1, 2, 3\}$ be the set and $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ be a permutation of $\Omega$ such that

$$P_1(1) = 2,\ P_1(2) = 1,\ P_1(3) = 3.$$

Then $$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ P_1(1) & P_1(2) & P_1(3) \end{pmatrix}$$

The element in the upper row form the domain of the permutation and the elements in the lower row constitute the image set of the permutation $P_1$. Al the permutations over the set $\Omega$ form a group under the binary operation of functional composition (from left to right). Such a group of all permutations over the set of n elements is called the symmetric group of degree n and is denoted by $S_n$. From our previous knowledge we know that $S_n$ is of order $\lfloor n$.

**Example.** $S_3$ is a nonabalian group of order 6.

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = P_0,\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_1,\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_2,\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_3, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_4,\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5 \right\}$$

It is the symmetric group of degree 3. $S_3$ has the following multiplication Table.

| · | $P_0$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|---|
| $P_0$ | $P_0$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
| $P_1$ | $P_1$ | $P_0$ | $P_5$ | $P_4$ | $P_3$ | $P_2$ |
| $P_3$ | $P_2$ | $P_4$ | $P_0$ | $P_5$ | $P_1$ | $P_3$ |
| $P_3$ | $P_3$ | $P_5$ | $P_4$ | $P_0$ | $P_2$ | $P_1$ |
| $P_4$ | $P_4$ | $P_2$ | $P_3$ | $P_1$ | $P_5$ | $P_0$ |
| $P_5$ | $P_5$ | $P_3$ | $P_1$ | $P_2$ | $P_0$ | $P_4$ |

$P_0$ is the identity permutation of $S_3$.

$P_1^{-1} = P_1$, $P_2^{-1} = P_2$, $P_3^{-1} = P_3$, $P_4^{-1} = P_5$, $P_5^{-1} = P_4$.

Verification of Associative property is left for the reader.

Further $P_1P_3 = P_4 \neq P_3P_1 = P_5$

Hence $S_3$ is a non-abelian group.

It may be noted that $S_3$ is the smallest non-abelian group.

Similarly the group $S_n$ of all permutations over the set $\{1, 2, 3, \cdots, n\}$ of n elements can be constructed.

**Remarks.** Keeping in view the role of the groups of permutations, a separate title is given to that in 2.6. of this chapter.

### 2.3.8. THEOREM

Let M(G) be the set of all mappings over a set G. Then M(G) forms a semi-group under the binary operation of composition of mappings.

**PROOF** Let $\phi, \psi, \theta \in M(G)$. Then

$\phi\circ\psi, \psi\circ\phi, \theta\circ\phi, \phi\circ\theta, \psi\circ\theta, \theta\circ\psi$ are mappings over G by Cor. of theorem 2.1.9

Further,

$$(\phi \circ \psi) \circ \theta\ (g) = \phi \circ \psi\ (\theta\ (g))$$

$$= \phi\ (\psi\ (\theta(g)))$$

$$= \phi \circ (\psi \circ \theta)\ (g)$$

for every $g \in G$.

Thus $(\phi \circ \psi) \circ \theta = \phi \circ (\psi \circ \theta), \quad \forall \psi, \theta, \phi \in M(G)$

Hence M(G) is a semi-group.

### 2.3.9. THEOREM

Let G be a non-empty set and $\bar{M}$ (G) be the set of all mappings which are (1 – 1) and onto G. Then $\bar{M}$ (G) forms a group under the binary operation of composition of mappings.

**Proof.** From theorems 2.3.5, and 2.1.10 and Cor. of theorem 2.1.9. the proof follows,

## 2.3.10. SOME IMPORTANT CONCEPTS

### (a) ORDER OF THE GROUP G

The number of elements in a group G, as a set, is called the order of the group G.

For example, the group $G = \{\pm 1, \pm i\}$ is of order 4 while all the real, rational and complex numbers are infinite groups under addition.

### (b) POWER OF AN ELEMENT OF A GROUP

Let G be a group under the binary opertion '·' . If $g \in G$ then,

$$g \cdot g = g^2,\ g \cdot g \cdot g = g^3,\ g \cdot g \ldots \text{(n times)} = g^n \text{ etc.}$$

are called the powers of g in G. In $g^n$, n, positive integer, is the index of the power of g. They observe the following index laws:

(i) $g^m \cdot g^n = g^{m+n}$

(ii) $(g^m)^n = g^{mn} = (g^n)^m$

(iii) $(g^n)^{-1} = g^{-n} = (g^{-1})^n$

(iv) $g^m / g^n = g^{m-n}$ $(m > n)$, when m, n, are positive integers.

## (c) ORDER OF AN ELEMENT OF A GROUP

Let g be an element of a group G. If n is the least positive integer such that $g^n = e$, the identity of the group G, then n is said to be the finite order of g in G. If there exists no such n then g is of infinite order.

**FOR EXAMPLE.** In the group $\{1, -1, i, -i\} = G$, since,

$$(-1)^2 = -1$$

and

$$(i)^4 = 1,$$

therefore, 2 and 4 are respectively, the orders of –1 and i in G.

**It may be noted that**

(1) If n is the order of an element g in a group G, then $g^{2n} = e = g^{3n} = \ldots = g^n$. Thus if $g^k = e$ and n is the order of g, then n divides $k = t \cdot n$, $t \in Z^+$.

(2) The identity element is the only element in group G, which is of order 1.

(3) An infinite group may have elements of finite and infinite order.

(4) A finite group has only the elements of finite order.

### 2.3.11. THEOREM

Let g be an element (non-identity) of a goup G. Then g is its own inverse if and only if g is of order 2.

**PROOF** Suppose $g \in G$ and g is its own inverse. Then

$g = g^{-1} \Rightarrow g \cdot g = g^{-1} \cdot g.$

$\Rightarrow g^2 = e. \Rightarrow$ 2 is the order of g.

Conversely, suppose g is of order 2. Then $g^2 = e$

$\Rightarrow g \cdot g = e$

$\Rightarrow (g \cdot g) \cdot g^{-1} = e \cdot g^{-1}$ (closure property)

$\Rightarrow g \cdot (g \cdot g^{-1}) = g^{-1}$ (associative property)

$\Rightarrow g \cdot e = g^{-1}$ (inverse property)

$\Rightarrow g = g^{-1} \Rightarrow$ g is its own inverse. (identity property)

### 2.3.12. THEOREM

Let $g_1, g_2$ be two elements of a finite group G. Then the order of $g_1 \cdot g_2$ is the same as that of $g_2 \cdot g_1$.

**PROOF** Since the group G is finite, therefore the element $g_1 \cdot g_2$ is of finite order n, say. Then

$$(g_1 \cdot g_2)^n = e$$

$$\Rightarrow (g_1 \cdot g_2)(g_1 \cdot g_2) \ldots\ldots (g_1 \cdot g_2) \text{ (n – times)} = e$$

$$\Rightarrow g_1 \underbrace{(g_2 \cdot g_1) \ldots\ldots (g_2 \cdot g_1)}_{\text{(n – 1) – times)}} \cdot g_2 = e \qquad \text{(associative property)}$$

$$\Rightarrow g_1^{-1} \cdot g_1 \underbrace{(g_2 \cdot g_1) \ldots\ldots (g_2 \cdot g_1)}_{\text{(n – 1) – times}} g_2 \cdot g_1 = g_1^{-1} \cdot e \cdot g_1 \qquad \text{(closure property)}$$

$$\Rightarrow (g_2 \cdot g_1) \ldots\ldots (g_2 \cdot g_1) \text{ (n–times)} = e. \qquad \text{(identity property)}$$

$\Rightarrow$ The order d of $(g_2 \cdot g_1)$ in G is a factor of n i.e. d/n.

Similarly if, $d(\neq n)$ is the order of $(g_2 \cdot g_1)$ then $(g_1 \cdot g_2)^d = e$, which implies that n/d. Consequently d = n. Hence the order of $g_2 \cdot g_1$ is the same as that of $g_1 \cdot g_2$.

**EXAMPLE** Let G be a group and $g \in G$. If the order of g is n then;

(i) The order of $g^{-1}$ is n.

(ii) The order of $x.g.x^{-1}$ is n.

**SOLUTION**

(i) If the order of g is n, then there exists n, the least positive integer, such that $g^n = e$.

$$e = g^n \cdot g^{-n} = g^n \cdot (g^n)^{-1} = (g^n)^{-1} = (g^{-1})^n$$

$\Rightarrow$ The order d of $t = g^{-1}$ is a factor of n.i.e. d/n.

If $(g^{-1})^d = e$ $(n \neq d)$ then $g^d = e$. But the order of g is n,

therefore n/d. Hence d = n and the order of $g^{-1}$ is the same as that of g.

(ii) Since $g^n = e$,

therefore,

$$(t)^n = (x\,gx^{-1})^n = x\,g^n\,x^{-1} = x\,ex^{-1} = xx^{-1} = e.$$

$\Rightarrow$ The order d, of $t = x\,gx^{-1}$, is a factor of n.
If d is the order of $xgx^{-1}$ then $(xgx^{-1})^d = xg^d x^{-1} = e$.
This implies that $g^d = x^{-1}ex = x^{-1}x = e$.
Thus n, the order of g, must divide d.
From d/n and n/d, it implies that $d = n$.

**EXAMPLE** In a Group G, $(g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$, for every $g_1, g_2 \in G$.
Suppose that x in G is the inverse of $g_1 \cdot g_2$. Then,

$$\begin{aligned}
& x(g_1 \cdot g_2) = e \\
\Rightarrow\ & xg_1 g_2 g_2^{-1} = eg_2^{-1} \\
\Rightarrow\ & (xg_1)\,e = g_2^{-1} \\
\Rightarrow\ & xg_1 = g_2^{-1} \\
\Rightarrow\ & xg_1 g_1^{-1} = g_2^{-1} \cdot g_1^{-1} \\
\Rightarrow\ & xe = g_2^{-1} \cdot g_1^{-1} \\
\Rightarrow\ & x = g_2^{-1} \cdot g_1^{-1}
\end{aligned}$$

Similar result follows if we suppose that $g_1 g_2\, x = e$

**NOTE** Such a rule can be extended to the product of any finite number of elements of a group.

i.e. $(g_1 \cdot g_2 \ldots . g_{n-1} \cdot g_n)^{-1} = g_n^{-1} \cdot g_{n-1}^{-1} \ldots . g_2^{-1} \cdot g_1^{-1}$.

## 2.3.13. GENERATING SET OF A GROUP

Let G be a group and $H = \{h_1, h_2, \cdots, h_n \cdots\}$ be a subset of G. If all the elements of G can be obtained by mutual multiplication of the elements of H and H is smallest such subset, then G is said to be a group generated by H. It is written as.

$$G = < h_1, h_2, \ldots . h_n, \ldots . > = < H >.$$

The elements of a group can be obtained by more than one subsets of G. The group G itself as a set, produces trivially the group G. If there is no proper subset of G which generates G then G itself is the generating set of G.

## 2.3.14. CYCLIC GROUP

A group G is said to be a cyclic group if all of its elements can be obtained, or generated, as powers of a single element g of G. The element $g \in G$ is called the generator of G and is denoted by $G = < g >$.

If G is a finite cyclic group then there exists a positive integer n, and a generator g in G, such that $g^n$ is the identity element of the group G. If there exists no such n then G is infinite cyclic group. If G is a cyclic group of order 7 then,

$$G = \{ <g> : g^7 = e \}$$

$$= \{ g, g^2, g^3, g^4, g^5, g^6, g^7 = e \}$$

**FOR EXAMPLE** The group $G = \{1, -1, i, -i\} = <i>$ is a cyclic of order 4 as

(i) $i^1 = i$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$, the identity of G.

**NOTE** (i) Any cyclic group of order n contains at least one element of order n.

(ii) If $G = <g>$, then $G = <g^{-1}>$.

i.e. If g is a generator of G then the inverse $g^{-1}$ of g is also a generator of G.

(iii) If G is a cyclic group of order n and g is a generator of G then $g^r$ is also a generator of G if $(r, n) = 1$, $r < n$.

**EXAMPLES.**

1. The group $U_n$, of all nth roots of unity is cyclic under the binary operation of multiplication of complex numbers

$$U_n = \left\{ < e^{\frac{2\pi m i}{n}} >, : m < n, m, n \text{ are positive integers and } (m, n) = 1 \right\}$$

2. The group Z of all integers under addition is a cyclic infinite group, where

$$Z = <1> = <-1>.$$

**2.3.15. THEOREM**

Every finite cyclic group is abelian.

**PROOF** Let G be a cyclic group of order n. Then $G = <g>$, say,

then $g^{r_1}, g^{r_2} \in G$,

where $1 \leq r_1, r_2, < n$.

Then $g^{r_1} \cdot g^{r_2} = g^{r_1 + r_2}$ (by definition)

$= g^{r_2 + r_1}$, (commutative property of positive integers)

which proves the theorem.

Every abelian group many not essentially be a cyclic group.

Note that

$$G = \{\langle g_1, g_2 \rangle : g_1^2 = g_2^2 = (g_1 \cdot g_2)^2 = e\}$$

is a group of order 4 which is abelian but not cyclic

## EXERCISE 2.3

1. Prove or disprove the following statements :

   (i) The set $\{ 2^n : n \in Z \}$ forms a group under the binary operation of multiplication.

   (ii) The set $\{ x\underline{i} + 0\underline{j} : x \in R \}$ of vectors in the Euclidean plane forms a group under vector addition.

   (iii) The set $\{ 0\underline{i} + y\underline{j} : y \in R \}$ of vectors in the Euclidean plane forms a group under vector addition.

   (iv) The set $\{0, 1\}$, under the product defined by, $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 0 = 0$ $1 \cdot 1 = 1$, forms a group.

   (v) The set $\{a + b\sqrt{2} : a, b \in Q$, not both zero$\}$ under the binary operation of multiplication forms a group.

   (vi) The set Z of all integers under the binary operation $a \cdot b = 2a + 3b$ for every $a, b \in Z$, forms a group.

   (vii) The set Q of all rationals under the binary operation $a \cdot b = a + b - ab$ forms a group.

   (viii) The set $\{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in Q$ not all $a, b, c$ are zero simultaneously$\}$ under the operation of multiplication forms a group.

2. Which of the groups in the above $Q_1$ are abelian?

3. Construct the multiplication table for the additive group $Z_6$ of residue classes mod 6. Give the identity element and order of each element of $Z_6$. Show that, under multplication, $Z_6$ does not form a group.

4. If, in a group, $b = x \cdot a \cdot x^{-1}$, show that $b^2 = x \cdot a^2 \cdot x^{-1}$ and hence show that the elements $a$ and b are of the same order.

5. Give all the single generators of the cyclic groups $C_5$, $C_8$ and $C_{12}$ of orders 5, 8 and 12 respectively.

6. In the group $D_6 = \{ < a, b > : a^3 = e = b^2, b\,a\,b = a^2 \}$,
   (i) Find the orders of $ba^2$, $ab$.
   (ii) Find the inverses of b, $a^2b$ and $ab$.

7. Let $C_7$ be the cyclic group of order 7. Find all the generators of the group $C_7$.

8. If, all the non-identity elements of a group G are of order 2, then prove that G is an abelian group.

9. The set $\left\{ \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} : x \in R^* \right\}$ of matrices forms an abelian group under matrix multiplication.

10. Prove that, the set of all mappings $\phi_d : R \to R.$, defined by $\phi_d(x) = x + d$, for every real number d, forms a group under composition.

11. Let $(G, *)$ and $(G', o)$ be two groups. Prove that the set $G \times G' = \{(g, g') : g \in G, g' \in G'\}$ forms a group under the binary operation of multiplication defind by, $(g_1, g_1') \cdot (g_2, g_2') = (g_1 * g_2, g_1' \circ g_2')$,
    for all $(g_1, g_1'), (g_2, g_2') \in G \times G'$.

12. Prove that, all the symmetries of an equilateral triangle, form a group.

13. Prove that, all the symmetries of a square from a group.

14. Which of the following groups are cyclic?
   (i) Additive group of reals.
   (ii) Additive group of rationals.
   (iii) Multiplicative group of positive rationals.

15. let G be a finite cyclic group of order n. For each factor m of n, prove that G contains a subset which forms a group, under the same operation as that of G, of order m.

16. Prove that, a cyclic group G has no proper subset which forms a group, under the same operation as that of G, if its order is prime or 1.

17. If m, n are relatively prime, then prove that the groups $C_m \times C_n$ and $C_{m.n}$ have same order.

18. Prove that all the even integers form an additive abelian group.

(i) $T_1 . T_2$ (ii) $T_1^{-1}$ (iii) $T_2^{-1}$ (iv) $T_1^{-1} . T_2$

(v) $T_1^{-1} . T_1^{-1}$ (vi) $T_2 . T_1^{-1}$ (vii) $T_1^{-2}$ (viii) $T_2^2$

(ix) The orders of $T_1$, $T_2$, $T_1 . T_2$ and $T_2 . T_1$.

19. If G is a group and $a \in G$ such that $a^2 = a$, then prove that a is the identity of G.

20. If, a group G has three elements then show tht G is abelian.

21. If, every element of the group G is its own inverse, then show that G is abelian.

22. Prove that every abelian group of prime order is cyclic

23. If, in a group G, $a^5 = e$ and $ab\,a^{-1} = b^2$, for $a, b \in G$, then find the order of b.

24. If, in a group G, $x \cdot a = a \cdot x$, then prove that $x \cdot a^2 = a^2 \cdot x$ and $x^2 \cdot a = a \cdot x^2$.

## 2.4 GROUP HOMOMORPHISMS

The mappings from one group $(G, \cdot)$ into another group $(G', *)$ which preserve the group structure, are called group homomorphisms. Their basic properties, which enhance the importance of this concept for the study of groups, are discussed under this heading.

### 2.4.1 DEFINITION

Let $(G, \cdot)$ and $(G', *)$ be two groups. A map $\psi : G \longrightarrow G'$, from G into G' is said to be a group homomorphism, if,

for every $g_1, g_2$ in G and their respective images $\psi(g_1), \psi(g_2)$ in G',

$$\psi(g_1 . g_2) = \psi(g_1) * \psi(g_2).$$

It further defines that,

(i) $\psi$ is called an endomorphism of G, if $G' = G$.

(ii) $\psi$ is called an epimorphism if $\psi$ is onto G'.

(iii) $\psi$ is called a monomorphism if $\psi$ is $(1 - 1)$.

(iv) $\psi$ is called an isomorphism if $\psi$ is $(1 - 1)$ and onto G' both.

(v) An endomorphism $\psi$ of a group G is called an automorphism of G if $\psi$ is $(1 - 1)$ and onto G.

## EXAMPLES

**1.** A map $\psi : (R, +) \longrightarrow (R^*, \cdot)$ defined by $\psi(m) = e^m$, from the group $(R, +)$ into the group $(R^*, \cdot)$, is a homomorphism, because,

$$\forall m_1, m_2 \in R, \quad \psi(m_1 + m_2) = e^{m_1 + m_2} = e^{m_1} \cdot e^{m_2} = \psi(m_1) \cdot \psi(m_2)$$

Since $\psi$ is $(1 - 1)$ therefore $\psi$ is a monomorphism. As $\psi$ is not onto$(R^*, \cdot)$, therefore $\psi$ is not an epimorphism and consequently $\psi$ is not an isomorphism.

**2.** Map $\psi : (R, +) \longrightarrow (R,+)$, from R into R, defined by,

$$\psi(m) = 2m, \quad \forall m \in R,$$

is an endomorphism of R because,

$$\forall m_1, m_2 \in R, \qquad \psi(m_1 + m_2) = 2(m_1 + m_2)$$
$$= 2m_1 + 2m_2$$
$$= \psi(m_1) + \psi(m_2)$$

which is also a monomorphism, an epimorphism and hence an automorphism of R.

**3.** Map $\psi : (R^*, \cdot) \longrightarrow (R^*, \cdot)$, defined by $\psi(m) = 2m$,
is not an endomorphism of $R^*$ because,

$$\psi(m_1 \cdot m_2) = 2m_1 \cdot m_2.$$
$$\neq 2m_1 \cdot 2m_2 = \psi(m_1) \cdot \psi(m_2)$$

**4.** Map $\psi : (R \times R, +) \longrightarrow (V_2(R),+)$ defined by,

$$\psi(x, y) = x\underline{i} + y\underline{j}$$

is a homomorphism, because, $\psi(x', y') = x'\underline{i} + y'\underline{j}$

and
$$\psi[(x, y) + (x', y')] = \psi[(x + x', y + y')]$$
$$= (x + x')\underline{i} + (y + y')\underline{j}$$
$$= x\underline{i} + x'\underline{i} + y\underline{j} + y'\underline{j}$$
$$= (x\underline{i} + y\underline{j}) + (x'\underline{i} + y'\underline{j})$$
$$= \psi(x,y) + \psi(x', y')$$

$\psi$ is an isomorphism. The proof is left to the reader as an exercise.

5. Let $(G, \cdot)$ be a group. For a fixed element $g \in G$. A mapping $\phi_g$ defined by $\phi_g(x) = g \cdot x \cdot g^{-1}, \forall x \in G$

is an automorphism of G.

Take $x_1, x_2 \in G$, then $\phi_g(x_1) = \phi_g(x_2)$

$$\Rightarrow \quad g x_1 g^{-1} = g x_2 g^{-1}.$$

$$\Rightarrow \quad x_1 = x_2 \text{ (by the cancellation laws).}$$

This holds for all $x_1, x_2 \in G$. Hence $\phi_g$ is a $(1-1)$ mapping over G.

For every $t \in G$, these exist $x \in G$ such that

$$\phi_g(x) = g x g^{-1} = t$$

$$\Rightarrow x = g^{-1} t g$$

which proves that $\phi_g$ is onto G.

Further, $\phi_g(x_1 \cdot x_2) = g(x_1 \cdot x_2) g^{-1}$

$= g x_1 (g^{-1} g) x_2 g^{-1}$. (identity property)

$= (g x_1 g^{-1}) \cdot (g x_2 g^{-1})$ (associative property)

$= \phi_g(x_1) \cdot \phi_g(x_2)$ (definition)

Thus $\phi_g$ is a $(1-1)$ homomorphism onto G and hence an automorphism of G.

Such an automorphism is associated with every element of the group and is called conjugation over G. If $g = e$, then $\phi_e$, is the identity automorphism of G.

6. Let $(G, \cdot)$ be a group. A mapping $\psi : G \longrightarrow G$, over G defined by

$$\psi(g) = g^{-1}, \ \forall g \in G,$$

is an automorphism of G if G is abelian.

Take $g_1, g_2 \in G$. Then $\psi(g_1) = g_1^{-1}$ and $\psi(g_2) = g_2^{-1}$.

The mapping $\psi$ is evidently $(1-1)$ and onto.

Thus $\psi(g_1 \cdot g_2) = (g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$

$= g_1^{-1} \cdot g_2^{-1}$ (if G is abelian)

$= \psi(g_1) \cdot \psi(g_2)$

Hence $\psi$ is an automorphism of an abelian group G.

**7.** Let $\phi_g$ be a mapping associated with an element g of a group G, defined by

$$\phi_g(x) = g \cdot x$$

Then $\phi_g$ is (1 – 1) and onto G.

Take $x_1, x_2 \in G$, then $\phi_g(x_1) = g \cdot x_1$

$\phi_g(x_2) = g \cdot x_2$

$\phi_g(x_1) = \phi_g(x_2) \Rightarrow g \cdot x_1 = g \cdot x_2 \Rightarrow x_1 = x_2$ (cancellation law),

which implies that $\phi_g$ is (1 – 1)

$\phi_g$ is evidently onto G because, for every $g \cdot x = t \in G$.

There exists $x = g^{-1} \cdot t \in G$, i.e, $\phi_{g^{-1}}(g \cdot x) = x$

and $\phi_{g^{-1}}(g \cdot e) = e \in G$.

But $\phi_g$ is not an endomorphism of G, because

$\phi_g(x_1 \cdot x_2) = g \cdot (x_1 \cdot x_2)$

$\neq (gx_1) \cdot (gx_2) = \phi_g(x_1) \cdot \phi_g(x_2)$

Similarly $\phi_g(x) = x \cdot g$ is a (1 – 1) and onto mapping over the group G.

**8.** Let $M_2(R)$ be the multiplicative group of non-singular 2 x 2 matrices with real entries. We define a map $\delta : M_2(R) \longrightarrow R^*$ by

$$\delta(A) = |A|.$$

Then $\delta$ is a homomorphism, because

If $\delta(A) = |A|$ and $\delta(B) = |B|$

Then $\delta(A \cdot B) = |A \cdot B| = |A| \cdot |B|$

$= \delta(A) \cdot \delta(B)$.

$\delta$ is not $(1-1)$, because equality of the determinants of matrices does not imply the equality of matrices. Guess, whether $\delta$ is onto?

### 2.4.2 DEFINITION

Two given groups $(G,\cdot)$ and $(G',\circ)$ are said to be isomorphic to each other if there exists an isomorphism between G and G'.

They are written as $G \cong G'$. and read as "the group G is isomorphic to the group G'."

Two isomorphic groups have the same structure. The only difference between them is that of the nature of their elements and the binary operation over G and G'.

### 2.4.3 THEOREM

The relation '$\cong$' of isomorphism over the set of all the groups is an equivalence relation.

**PROOF** Let $G_1$, $G_2$ and $G_3$ be three groups. Then $G_1 \cong G_1$ by the identity isomorphism.

For symmetry, suppose $G_1 \cong G_2$. Than there exists an isomorphism

$\psi : G_1 \longrightarrow G_2$. Since $\psi$ is a $(1-1)$ and onto homomorphism from $G_1$ onto $G_2$ therefore

$\psi^{-1} : G_2 \longrightarrow G_1$ is an isomorphism.

Thus $G_2 \cong G_1$.

For transitivity, suppose, $G_1 \cong G_2$ and $G_2 \cong G_3$. Since $G_1$ is isomorphic to $G_2$ and $G_2$ is isomorphic to $G_3$ therefore, there exist, isomorphisms $\psi_1$ and $\psi_2$ such that $\psi_1 : G_1 \longrightarrow G_2$ and $\psi_2 : G_2 \longrightarrow G_3$.

We know by theorem 2.3.10 that $\psi_2 \circ \psi_1 : G_1 \longrightarrow G_3$ is an isomorphism from $G_1$ into $G_3$. It proves that $G_1 \cong G_3$.

Hence the relation '$\cong$' is an equivalence relation over the set of all groups.

### 2.4.4 THEOREM

Every finite cyclic group $C_n$ of order n is isomorphic to the group $U_n$ of all n, nth roots of unity.

**PROOF** Suppose $C_n = \{<g> : g^n = e\}$,

and $U_n = \left\{ e^{\frac{2\pi m i}{n}} : m = 0, 1, 2, \ldots\ldots n-1 \right\}$

A map, $\theta : C_n \longrightarrow U_n$ is defined by

$$\theta(g^{r_1}) = e^{\frac{2\pi r_1 i}{n}}$$

and $$\theta(g^{r_2}) = e^{\frac{2\pi r_2 i}{n}}$$

Then $$\theta(g^{r_1} \cdot g^{r_2}) = \theta(g^{r_1+r_2}) = e^{\frac{2\pi (r_1 + r_2) i}{n}}$$

$$= e^{\frac{2\pi r_1 i}{n} + \frac{2\pi r_2 i}{n}}$$

$$= e^{\frac{2\pi r_1 i}{n}} \cdot e^{\frac{2\pi r_2 i}{n}}$$

$$= \theta(g^{r_1}) \cdot \theta(g^{r_2})$$

It proves that $\theta$ is a homomorphism from $C_n$ into $U_n$.

Evidently, $\theta$ is (1 – 1) and onto $U_n$ because, $\theta(g^{r_1}) = e^{\frac{2\pi r_1 i}{n}} = e^{\frac{2\pi r_2 i}{n}} = \theta(g^{r_2})$

$$\Rightarrow e^{\frac{2\pi (r_1 - r_2)^2}{n}} = 1$$

$$\Rightarrow r_1 - r_2 = 0$$

$$\Rightarrow r_1 = r_2 \Rightarrow g^{r_1} = g^{r_2}$$

Hence $\theta$ is an isomorphism and $C_n \cong U_n$.

**COR.** All the finite cyclic groups of order n are isomorphic by theorem 2.4.3.

### 2.4.5 THEOREM

Every infinite cyclic group is isomorphic to the group (Z,+) of all integers under addition.

**PROOF** The group $(Z,+) = \{n : n \in Z\}$ of all integers. Let $C_\infty$ be the infinite cyclic group generated by an element $g \in C_\infty$. Then $C_\infty = \{<g>; g, g^2, g^3, g^n, \ldots\}$. Define a map $\psi : C_\infty \longrightarrow Z$, by $\psi(g^n) = n$.

Since $g^n \neq g^m$ in $C_\infty$, therefore,

$$g^m = g^n \Rightarrow m = n.$$

and $\quad m = n \Rightarrow g^m = g^n,$

which prove that $\psi$ is a $(1-1)$ mapping.

$\psi$ is onto, because, for every $r \in Z$, $g^r \in C_\infty$.

Further $\quad \psi(g^{m_1} . g^{m_2}) = \psi(g^{m_1+m_2}) = m_1 + m_2 = \psi(g^{m_1}) + \psi(g^{m_2}).$

Hence $\psi$ is an isomorphism and the groups $C_\infty$ and $(Z,+)$ are isomorphic to each other.

**COR.** All the infinite cyclic groups are isomorphic.

Let $G_1$, $G_2$ be two infinite cyclic groups. Then $G_1 \cong (Z,+)$ and $G_2 \cong (Z,+)$ and by theorem 2.4.3 $G_1 \cong G_2$.

### 2.4.6 DEFINITION

Let $\psi : (G, \cdot) \to (G', \cdot)$ be a group homomorphism. Then the subset $\{g \in G : \psi(g) = e'\}$ of G, consisting of all the elements of G whose image is the identity of G′, is called the Kernel of the homomorphism $\psi$. It is denoted by Ker $\psi$.

### 2.4.7 THEOREM

Let $(G, \cdot)$ and $(G', \circ)$ be two groups and $\psi : G \longrightarrow G'$ be a homomorphism from G onto G′.

Then

(i) $\psi(e)$ is the identity of G′, if e is the identity of G.

(ii) $\psi(a^{-1}) = [\psi(a)]^{-1}$, i.e.

If $a^{-1}$ is the inverse of $a$ in G then $[\psi(a)]^{-1}$ is the inverse of $\psi(a)$ in G′.

**PROOF**

(i) We know that, $\forall g \in G$,

$\psi(g) = \psi(g.e) = \psi(g) \circ \psi(e)$, since $\psi$ is a homomorphism.

Since $\psi$ is onto, therefore $\forall \psi(g) \in G'$,

$$\psi(g) \circ \psi(e) = \psi(g)$$

Similarly $\psi(e) \circ \psi(g) = \psi(g)$

Hence $\psi(e)$ is the identity of $(G', \circ)$

(ii) $\psi(e) = \psi(g \cdot g^{-1}) = \psi(g^{-1}.g), \forall g \in G.$

$= \psi(g) \circ \psi(g^{-1}) = \psi(g^{-1}) \circ \psi(g)$ $\because \psi$ is a homomorphism

As $\psi(e)$ is the identity of $G'$, therefore,

$$\psi(g^{-1}) = [\psi(g)]^{-1}, \quad \forall g \in G.$$

It completes the proof of the theorem.

### 2.4.8 THEOREM

Let $\phi : G \to G'$ be a group homomorphian. Then $\phi$ is a monomorphism if and only if Ker $\phi = \{e\}$.

**PROOF**

Suppose that $\phi$ is a group-homomorphism such that Ker $\phi = \{e\}$. If $g_1, g_2 \in G$ and their images under $\phi$ coincide, then

$$\Rightarrow \quad \phi(g_1) = \phi(g_2)$$

$$\Rightarrow \quad \phi(g_1) \cdot \phi(g_2)^{-1} = e'$$

$$\Rightarrow \quad \phi(g_1) \cdot \phi(g_2^{-1}) = e'$$

$$\Rightarrow \quad \phi(g_1 \cdot g_2^{-1}) = e'$$

$$\Rightarrow \quad g_1 \cdot g_2^{-1} \in \text{Ker } \phi.$$

Since Ker $\phi = \{e\}$, therefore

$$g_1 \cdot g_2^{-1} = e \Rightarrow g_1 = g_2 = \varphi \text{ is } (1-1)$$

Considering if $\varphi$ is $(1-1)$, and $e \neq g \in \text{Ker } \varphi$, then $\varphi(g) = e = \varphi(e)$ and hence it replies that $g = e$, since $\varphi$ is $(1-1)$. Hence the proof.

### 2.4.9 THEOREM

Let $\psi : G \longrightarrow G'$ be a group homomorphism.

If $g \in G$ is of order n then the order of $\psi(g)$ is a factor of n.

**PROOF** If $g \in G$ is of order n then $g^n = e$,

and $\psi(e) = \psi(g^n) = \psi(g \cdot g \cdots g)$ (n times)

$= \psi(g) \cdot \psi(g) \ldots \psi(g)$ (n-times)

$= [\psi(g)]^n$

Since $\psi(e)$ is the identity of G', therefore, $[\psi(g)]^n$ = the identity of G'.

Hence the order of $\psi(g)$ is a factor of n.

**Cor.** $\psi(g)$ is of order equal to the order n of g if $\psi$ is a monomorphism.

## EXERCISE 2.4

1. Prove or disprove the following statements.

   (i) The identity map over a given group G is an endomorphism of G.

   (ii) The map $\theta : R^* \longrightarrow R^*$, defined by $\theta(m) = m^2$ is an endomorphism of $R^*$.

   (iii) Let $(G, \cdot)$ and $(G', \circ)$ be two groups. The mapping defined by $\phi(g) = e'$, for every $g \in G$, is homomorphism from G into G'.

   (iv) The mapping $\theta : G \longrightarrow G$. defined by $\theta(g) = g^{-1}$ is an endomorphism of G if G is abelian.

2. Prove that the following mappings are group homomorphisms. State also the type of the homomorphism;

   (i) $\theta : V_2(Z) \longrightarrow \mathbb{C}$ , defined by $\theta(m,n) = m + i\,n$

   (ii) $\theta : V_2(R) \longrightarrow \mathbb{C}$, defined by (a) $\theta(x,y) = x + i\,y$.

   (b) $\theta(x,y) = i\,y$

   (iii) $\theta : V_2(R) \longrightarrow V_2(R)$ defined by $\theta(x,y) = (x - y, x + y)$

   (iv) $\theta : V_2(R) \longrightarrow V_3(R)$ defined by $\theta(x,y) = (x - y, x + y, y)$

   (v) $\theta : V_3(R) \longrightarrow V_2(R)$ defined by $\theta(x,y,z) = (x - y, x + y)$

   (vi) $\theta : R \longrightarrow V_2(R)$ defined by $\theta(x) = (x,x)$.

3. Prove, whether the following mapping is a homomorphism ?

   $\theta : C_{10} \longrightarrow C_{10}$ defined by $\theta(a) = a^4$, when $a$ is a generator of $C_{10}$.

4. Prove that $\theta: V_2(R) \longrightarrow V_2(R)$ defined by $\theta(x,y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$ is an endomorphism of $V_2(R)$.

5. Let G be a group and $\theta$, $\phi$ be two homomorphisms over G, prove that if;
   (i) $\theta \circ \phi$ is an epimorphism then $\phi$ must be an epimorphism
   (ii) $\theta \circ \phi$ is a monomorphism then $\theta$ must be a monomorphism

6. If $\theta : C_n \longrightarrow H$ be a homomorphism from a cyclic group $C_n$ of order n into a group H, then $\theta(C_n)$ is a cyclic and its order is a factor of n.

7. Let $\phi$ and $\psi$ be two group homomorphisms. Then their products $\phi \circ \psi$ and $\psi \circ \phi$ are also homomorphisms.

8. Let $\psi : M_2(R) \longrightarrow R^*$ be a mapping, defined by $\psi(A) = |A|$. Then $\psi$ is a homomorphism.

9. Let $\theta : (\mathbb{C}, +) \longrightarrow (M_2(R), +)$ be a mapping defined by,

$$\theta(x + iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

   Prove that $\theta$ is an isomorphism.

10. Let $(G, \cdot)$ be a group and for a fixed g in G, $\psi_g$ be a mapping over G defined by $\psi_g(x) = g^{-1}xg$, $\forall x \in G$. Show that $\psi_g$ is an automorphism of G.

## 2.5 SUBGROUPS OF A GROUP

One of the techniques developed in the literature for the study of a group is based on the study of its subgroups. A subgroup is a subset of a group $(G, \cdot)$ which is a group itself under the same binary operation as that of G. The identity of the subgroup will be the same as that of the identity of G. Evidently, the associative property of the group descends to its every subset. The notation "$H \subseteq G$" is observed for, "H is a subgroup of a group G".

### 2.5.1 THEOREM

Let H be a subset of a group $(G, \cdot)$. Then H is a subgroup of G if and only if,

(i) For every $h_1, h_2 \in H$, $h_1 \cdot h_2 \in H$.

and (ii) For every $h \in H$, $h^{-1} \in H$.

**PROOF** Suppose H is a subgroup of $(G, \cdot)$. Then H is a group itself. Thus the properties (i) and (ii) follow by definition. Conversely, suppose (i) and (ii) hold in subset H of G. Then H is associative trivially. (i) and (ii) imply that the closure and the inverse properties hold in H. Further $h \cdot h^{-1} = e \in H$, by (i). Hence the identity property bolds in H. Thus H is a subgroup of G.

It is immediate, by combining (i) and (ii) together of the above theorem 2.5.1, that ;

**2.5.2 THEOREM**

Let H be a subset of group (G, ·). Then H is a subgroup of G iff, $\forall h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$.

**PROOF** Suppose H is a subgroup of a group (G, ·). Then H is a group. By the inverse and the closure property of a group it implies that $h_1 \cdot h_2^{-1} \in H$, $\forall h_1, h_2 \in H$.

Conversely, suppose $h_1 \cdot h_2^{-1} \in H$, $h_1, h_2 \in H$.

Then $h_1 \cdot h_1^{-1} = e \in H$.

$\forall h_1, h_2, h_3 \in H$, $(h_1 \cdot h_2) \cdot h_3 = h_1 \cdot (h_2 \cdot h_3)$ holds in H as it holds in G.

Then $e \cdot h^{-1} = h^{-1} \in H$, $\forall h \in H$ (inverse properly) and $h_1 \cdot (h_2^{-1})^{-1} = h_1 \cdot h_2 \in H$.

Thus the closure property holds in H. H is therefore a group under the same binary operation as that of G. As H is a subset of G, therefore H is a subgroup of G.

**Cor.** $gH = H$ if and only if $g \in H$.

It is important to note that if g is an element of a finite group G, then all the powers of g form a subset of G. This subset is always a subgroup of G since, $g^m.g^n = g^{m+n}$, $(g^n)^{-1} = g^{-n}$ and $g^0 = e$ (m,n are positive integers).

If g has infinite order, then subgroup of G consisting of all powers of g is infinite cyclic group. If g has finite order n then the subgroup of powers of g is the finite cyclic group $C_n$, of order n if $g^n = e$.

If G is abelian group then all of its subgroups ought to be abelian.

Let H be a subgroup of a group G. Then the sets

$H \,.\, H = H^2 = \{h_1 \,.\, h_2 : h_1, h_2 \in H\}$ and $H^{-1} = \{h^{-1} : h \in H\}$ are subsets of G.

**2.5.3 THEOREM**

If H is a subgroup of a group G, then (i) $H^2 = H$ and (ii) $H = H^{-1}$.

**PROOF** (i) Since H is a subgroup of G and $h_1, h_2 \in H$. Then $h_1 \cdot h_2 \in H^2$, which is also a member of H. Hence

$$H^2 \subseteq H.$$

Further, for every $h \in H$, $h = h \cdot e$, $e \in H$.

$$\Rightarrow h \in H^2 \quad \Rightarrow H \subseteq H^2.$$

Consequently $H = H^2$.

(ii) For every $h \in H$, $h^{-1} \in H^{-1}$, as $h^{-1} \in H$ therefore it implies that

$$H^{-1} \subseteq H.$$

$\forall h \in H$, $h^{-1} \in H$, as H a subgroup of G. Then by definition $(h^{-1})^{-1} \in H^{-1}$ but $(h^{-1})^{-1} = h \Rightarrow H \subseteq H^{-1}$.

Thus $H = H^{-1}$ consequently

### 2.5.4 THEOREM

Let H be a non-empty subset of a group G in which $H^2 = H$ and $H^{-1} = H$, then H is a subgroup of G.

**PROOF** Suppose H is a sub set of a group G in which $H^2 = H$ and $H^{-1} = H$.

$h, k \in H \Rightarrow h \cdot k \in H^2$ but $H^2 = H$ (supposition)

therefore $h \cdot k \in H$. Hence the closure property of H holds.

Further, for every $h \in H$, $h^{-1} \in H^{-1} = H$.

$$\Rightarrow h^{-1} \in H$$

By theorem 2.5.1 it follows that H is a subgroup of G.

### 2.5.5 THEOREM

Let H and K be two subgroups of a group G. Then $H \cap K$ is also a subgroup of G.

**PROOF** Suppose $g, h \in H \cap K$. Then h and g both are is H as well as in K. Since H and K are subgroups of G, therefore, by theorem 2.5.2, $gh^{-1} \in H$ and $gh^{-1} \in K$. It implies that $gh^{-1} \in H \cap K$.

Also $H \cap K$ is non empty because it contain the identity element of G.

Thus $H \cap K$ is a subgroup of G, by theorem 2.5.2.

It is important to note that $H \cup K$ need not be a group, if $g, h \in H \cup K$, there is no guarantee that they both (g and h) are either in H or in K, and hence we can say nothing about $g \cdot h^{-1}$.

## EXAMPLES

1. The set $\mathbb{C}$ of all complex numbers forms a group under addition. The set R of all real numbers is its subset which forms its subgroup. Q is a subgroup of R, and Z is a subgroup of Q. Hence

$$Z \subseteq Q \subseteq R \subseteq \mathbb{C}.$$

2. The set $\mathbb{C}^*$ forms a group under multiplication. $R^*$ is a subgroup of $\mathbb{C}^*$, $Q^*$ is a subgroup of $R^*$

Thus $Q^* \subseteq R^* \subseteq \mathbb{C}^*$.

3. As the set $\bar{M}_2(R)$ of all 2 x 2 non-singular matrices over the set R of all real numbers, forms a group under the binary operation of matrix multiplication, its subset consisting of matrices whose determinant is equal to 1 forms its subgroup. The subset D of $\bar{M}_2(R)$ consisting of all diagonal matrices forms another subgroup.

$$\text{i.e.} \quad D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} 0 \neq a, b \in R \right\}$$

4. If G is a group and g is a fixed element of G. Then the subset $\{x \in G : gx = xg\}$ of G consisting of those elements which observe the commutative property with g forms a subgroup of G, as,

if $gx_1 = x_1g$ and $gx_2 = x_2g$, then

$$(x_1 x_2^{-1})(g) = x_1(x_2^{-1}g) = x_1(gx_2^{-1}) = (x_1g)x_2^{-1} = (gx_1)x_2^{-1} = (g)(x_1 x_2^{-1})$$

Note that if x commutes with g then $x^{-1}$ also commutes with g, because

$$xg = gx \Rightarrow gx^{-1} = x^{-1}g.$$

The subgroup formed in Ex. 4 is called the centralizer of g in G and is denoted by $C_G(g)$. Such subgroup exists for every element of G.

If $g = e$, then $C_G(e) = G$.

If G is an abelian group then $C_G(g) = G$, $\forall g \in G$.

5. A subset of a group G, consisting of all those elements of G which observe the commutative property with every element of G, is called the centre of G, and is denoted by Z(G). The centre of a group G forms a subgroup of G.

As $e \in Z(G)$, therefore, Z(G) is always non-empty subset of G.

Take $z_1, z_2 \in Z(G)$ and $g \in G$

Then $(z_1 \cdot z_2^{-1})g = z_1 (z_2^{-1} g)$ (associative property)

$= z_1(g\, z_2^{-1})$ $(\because z_2 \in Z(G))$

$= (z_1\, g)z_2^{-1}$ (associative property)

$= (gz_1)z_2^{-1}$ $(\because z_1 \in Z(G))$

$= (g)\,(z_1\, z_2^{-1})$ (associative property)

which shows that $z_1 \cdot z_2^{-1} \in Z(G)$ and hence by theorem 2.5.2, Z(G) is a subgroup of G.

**Note that,**

(i) If G is abelian then Z(G) = G.

(ii) If G is non-abelian then $Z(G) \subset G$.

(iii) Z(G) is an abelian subgroup of G.

6. If $G : \{<a> : a^6 = e\}$
then (i) $H = \{e, a^3\} = \{<a^3> : a^6 = e\}$
and (ii) $K = \{e, a^2, a^4\} = \{<a^2> : a^6 = e\}$
are subgroups of G.

7. An element g in a group G generates a subgroup of order equal to the order of g in G.

## 2.5.6 THEOREM

Let $\psi : G \longrightarrow G'$ be a group homomorphism

Than (i) Ker $\psi$ is a subgroup of G.

(ii) If H is a subgroup of G then $\psi(H)$ is a subgroup of G'.

**PROOF** (i) We know that $Ker\psi = \{g \in G : \psi(g) = e'\}$

Ker$\psi$ is a non-empty subset of G because it contain at least the identity element of G.

Take $g_1, g_2 \in Ker\psi$. Then

$\psi(g_1) = e'$, $\psi(g_2) = e'$.

and $\psi(g_1 \cdot g_2^{-1}) = \psi(g_1) \cdot \psi(g_2^{-1})$, as $\psi$ is a homomorphism.

$= \psi(g_1) \cdot (\psi(g_2))^{-1}$, by theorem 2.4.8.

$= e' \cdot (e')^{-1} = e'$

Thus $g_1 \cdot g_2^{-1} \in Ker\psi$.

Hence Ker$\psi$ is a subgroup of G.

(ii) Since H is a subgroup of G, therefore, for every $h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$, by theorem 2.5.1.

As $\psi$ is a homomorphism, therefore, $\psi(h_1), \psi(h_2) \in \psi(H)$.

and $\psi(h_1 h_2^{-1}) = \psi(h_1) \cdot \psi(h_2^{-1})$

$= \psi(h_1) (\psi(h_2))^{-1} \cdot \in \psi(H)$, as $h_1 h_2^{-1} \in H$

for every $\psi(h_1), \psi(h_2) \in \psi(H) \subseteq G'$.

Hence $\psi(H)$ is a subgroup of G'.

## 2.5.7 COSETS

Let G be a group and H be a subgroup of G. For a fixed element g in G, the set $gH = \{g \cdot h : h \in H\}$ is defined to be a left coset of the group G with respect to its subgroup H which contains g. Similarly the right coset Hg is defined which contain g.

In general, for non-abelian groups left and right cosets do not coincide. The properties of the cosets, left or right are same and either may be used in this discussion. We follow here the study of left cosets of a group by any of its subgroups.

## 2.5.8 COSET FORMATION

Let G be a group and H be a subgroup of G. Pick $g_1 \in G$ such that $g_1 \notin H$. The left coset $g_1 H$ is formed. It may be noted that if $g_1 \in H$ then $g_1 H = H$. Now if $H \cup g_1 H \neq G$ then pick $g_2 \in G$ such that $g_2 \notin H \cup g_1 H$. The coset $g_2 H$ is formed. We continue this process till the whole group G is exhausted. Thus

$$G = \cup g_i H.$$

If G is a finite group, then the number of cosets are finite.

For example. $G = \{1, -1, i, -i\}$ be a group under multiplication and

$H = \{1, -1\}$ be a subgroup of G.

Then $1 \cdot H = H$ is one of the left cosets of G by H.

$i \notin H$. Therefore.

$$i \cdot H = \{i \cdot 1, i \cdot (-1)\} = \{i, -i\}$$

Since $H \cup iH = G$, therefore, two cosets are formed of a group G by its subgroup H.

### 2.5.9. QUOTIENT SET

The set of all cosets (left or right) of a group G by its subgroup H is said to form the quotient set and is denoted by G/H.

That is, $G/H = \{H, g_1H, g_2H, \ldots\}$

If G is finite and has n cosets by its subgroup H, then

$$G/H = \{H, g_1H, \ldots, g_{n-1}H\}$$

and n is the index of G by H, which is written as $[G : H] = n = |G| / |H|$

Note that the subgroup H of G is one of the cosets (left as well as right) of G by H.

### 2.5.10. THEOREM

**The left cosets of a group G by its subgroup H are either identical or disjoint.**

**PROOF** Let $g_1H$, $g_2H$ be two distinct left cosets of G by H. Suppose they are not disjoint. Then there exists an element $g \in g_1H \cap g_2H$.

As $g \in g_1H$, then it can be written as $g = g_1 \cdot h_1$, if $h_1 \in H$. ..... *(i)*

Similarly $g \in g_2H \Rightarrow g = g_2 h_2$, for some $h_2 \in H$. ..... *(ii)*

(i) and (ii) imply that $g_1 h_1 = g_2 h_2$

$\Rightarrow$ $g_1 = g_2(h_2 h_1^{-1})$

Since $h_2 \cdot h_1^{-1} \in H$, therefore $g_1 \in g_2H$ and $g_2h \in g_2H$, for every $h \in H$. Thus $g_1H \subseteq g_2H$.

Similarly from $g_2h_2 = g_1h_1$ it implies that $g_2H \subseteq g_1H$. It concludes that $g_1H = g_2H$. It means to say that $g_1H$ and $g_2H$ are identical. Which proves the assertion of the theorem via contradiction.

From the coset formation it is evident that the union of all the cosets produces the whole group G and their intersection is empty. Thus,

### 2.4.11. THEOREM

The cosets of a group G by any of its subgroups partition G into its subsets.

### 2.5.12. THEOREM

Each left coset of a group G by its subgroup H contains elements, equal in number, to the elements in H.

**PROOF** We establish a(1 – 1) correspondence between the elements of the sets $g_1H$ and H. The map $\psi : g_1H \to H$ defined by $\psi(g_1h) = h$ is (1 – 1), because if $\psi(g_1h_1) = h_1$ and $\psi(g_1h_2) = h_2$ then $h_1 = h_2 \Rightarrow g_1h_1 = g_1h_2$. It is evidently onto because for every $h \in H$, there exists $g_1h \in g_1H$.

Hence the mapping $\psi$ is bijective and sets H and $g_1H$ have same number of elements.

Consequently all the cosets have elements equal in number to the elements in H.

**Cor.** All the cosets of a finite group G w.r.t. its subgroup H have elements equal in number to the order $|H|$ of H.

## 2.5.13. LAGRANGE'S THEOREM

Let G be a finite group and H be its subgroup. Then the order of H divides the order of G.

**PROOF.** Since G is a finite group, therefore its subgroup H is finite and consequently G has finite number of cosets (left) by H, say n. Then

$$G = \{H, g_1H, \ldots, g_{n-1}H\}$$

where $$G = \cup g_iH$$

Let $g_i H$ and $g_j H$ be any two cosets of G by H. We define a map from $g_iH$ into $g_jH$, by

$$\phi : g_i H \to g_j H,$$

$$\phi(g_ih) = g_j h, \qquad \forall\ h \in H$$

Since $\phi(g_ih_1) = \phi(g_j h_2)$, therefore,

$$g_j h_1 = g_j h_2$$

$$\Rightarrow \quad h_1 = h_2$$

$$\Rightarrow \quad g_i . h_1 = g_i . h_2 \qquad \forall\ h_1, h_2 \in H$$

Therefore, $\phi$ is a one-to-one map.

$\phi$ is clearly a onto map, because, for every $g_j h \in g_j H$, these exists an element $g_ih \in g_iH$.

This way $\phi$ is a one-to-one correspondence and $|g_iH| = |g_j H|$ and hence all the cosets of G by H have the same number of elements equal in number to the order $|H|$ of H.

Since the relation '~' defined over G, by "$g_1 \sim g_2$ if and only if $g_1H = g_2H$" is reflexive, symmetric and transitive, therefore, '~' decomposes G into disjoint cosets

whose union is the whole of G. As each coset contains elements equal in number to the order of H, therefore

$$|G| = |H| \cdot n$$

or

$$n = \frac{|G|}{|H|},$$

where n is a positive integer, corresponding to number of cosets of G by H.

Hence the order of H divides the order of G.

**NOTE** The Lagrange's theorem revealed a marvellous result of the finite group theory. It opened the new doors for the study of the finite groups. The reader must be aware to note its converse that,

"For every factor d of the order n of a group G, there exists a subgroup of that order within G"

The connerse does not hold in general. The study of the finite groups in which the converse does hold for particular factors, is beyond the scope of this level of teaching.

**COR 1.** In a finite group G, the order of every element of G divides the order of G.

**PROOF** Let x in G be an element of order n. Then $x^n = e$ and all the powers $\{e, x, x^2, \ldots, x^{n-1}\}$ form a subgroup $\langle x \rangle$ of G which is of order n. By the lagrange's theorem n divides the order $|G|$ of G.

**COR 2.** In a finite group G of order m, $x^m = e$, for every element x in G.

**PROOF** If x is the identity element then it is trivial that $x^m = e$. If x is non-identity element, then it is of order d, say. Then d divides $|G|$, by Cor. 1.

and

$$\frac{|G|}{d} = k, \quad \text{say}$$

Then

$$x^{|G|} = x^{dk} = (x^d)^k = (e)^k = e \ .$$

**COR. 3.** If g is a finite group then all of its elements ought to be of finite order.

**COR. 4.** There is at the most one element of order 2 in a cyclic group of finite order.

**PROOF** If a cyclic group G is of odd order, then there is no element of order 2 in G.

If $|G|$ is even and x is the generator of G then $(x)^{|G|/2}$ is the only element of order 2 in G.

### 2.5.14. THEOREM

Let G be a finite cyclic group, then every subgroup H of the group G is also cyclic.

**PROOF** Since G is cyclic therefore, it can be generated by a single element of G.

If G is of order n, then

$$G = \{<a> : a^n = e\}$$

Suppose H is a subgroup of G and $a^r$ $(r < n) \in H$ and r is such smallest index of the power of $a$. All the powers of $a^r$ are also in H.

Thus $H = \{a^r, a^{2r}, \ldots, a^{tr}\}$, when $t \cdot r = n$.

$$\Rightarrow \quad r = \frac{n}{t} \quad \text{and} \quad a^{tr} = e = (a^r)^t$$

If $a^l$ is a general element of G then

$$l = r \cdot t + m, \quad 0 \leq m \leq r$$

$$a^m = a^l \cdot a^{-n} = a^l \cdot (a^t)^{-1} \in H \text{ only if } m = 0$$

Hence H contains no element other than powers of $a^r$. Thus H is generated by $a^r$ and is or order t. Hence H is cyclic. If (r, n) = 1 then H = G.

### 2.5.15. THEOREM

In an obelian group G, all the elements of finite order form a subgroup of G.

**PROOF** Let F be the subset of G consisting of the elements of finite order in G. If $x, y \in F$, then there exists positive integers m and n such that $x^m = e = y^n$ and consequently $(g^{-1})^n = y^{-n} = e$.

Since
$$(xy^{-1})^{m.n} = x^{m.n} (y^{-1})^{m.n}$$
$$= (x^m)^n \cdot (y^{-n})^m = e \cdot e = e$$

Therefore order of $xy^{-1}$ is a factor of $m \cdot n$ which is of finite order and hence $xy^{-1}$ is in F.

Hence F is a subgroup of abelian group G.

### 2.5.16. THEOREM

Let H be a subgroup of a group of a group G. Then the quotient group $G/H = \{g_1H = g_1 \in G\}$ forms a group, under the binary operation of multiplication $(g, H) \cdot (g, H) = (g_1 g_2)H$, if

$$g_1H = Hg_1, \text{ for all } g_i \in G.$$

**PROOF**

It is obvious that the multipliction is associative, i.e.,

$$\{(g, H) \cdot (g_2 H)\} \cdot (g_3 H) = (g_1 g_2 H) \cdot g_3 H$$
$$= (g_1 g_2) \cdot g_3 H = g_1 (g_2 g_3) H$$
$$= (g_1 H) \cdot \{(g_2 H) \cdot (g_3 H)\} \cdot g_2, g_2, g_3 \in G.$$

H is the identity element, because

$$H \cdot (g\,H) = (H\,g) \cdot H = (g\,H) \cdot H = gH^2 = gH$$

and $(gH)H = gH^2 = gH$.

Further more, since,

$$(gH)(g_1H) = H \Leftrightarrow gg_1H = H$$

$$\Leftrightarrow gg_1 = e \in G$$

$$\Leftrightarrow g_1 = g^{-1}$$

$g_1H = (gH)^{-1} = g^{-1}H$, is the inverse of $gH$ $(\forall\ g \in G)$.

Hence G/H is a group.

## EXERCISE 2.5

**1.** Prove or disprove the following statements:

(i) The centre of a group is its abelian subgroup.

(ii) The centre of the group $S_3$ contains only the identity element of $S_3$.

(iii) Let X be a complex of a group G. Then $C(X) = \bigcap_{x \in X} C_G(x)$ where

$$C(x) = \{g \in G : gx = xg\}$$

(iv) Let H be a subgroup of a group G and $g \in G$ then $gHg^{-1}$ and H are isomorphic to each other.

(v) Let H and K be two subgroups of a group G, then $H \cap K$ forms a subgroup of H as well as of K.

(vi) If H is a subgroup of a group G and K is a subgroup of H then K is a subgroup of G.

(vii) The symmetric group $S_3$ is a non-abelian group.

2. Prove that every subgroup of an abelian group is also abelian.

3. Let H be a subgroup of a group G. Then show that the subsets $a^{-1}Ha$, $aHa^{-1}$ of G form subgroups of G, for a fixed $a \in G$.

4. If, gH and Hg are left and right cosets of a group G by its subgroup H, then prove that gH and Hg have same number of elements in them.

5. Prove that a group G is finite if and only if it contains only finitely many subgroups.

6. List all the subgroups of $C_4$.

7. List all the subgroups of $C_8$.

8. Let H be a subgroup of a group G and the $g, h \in H$. The element g is related to h if $g \cdot h^{-1} \in H$. Prove that the relation is an equivalance relation on G.

9. If H is a subgroup of a group G and K, a complex with $K \subseteq H$, then show that

$$H \cdot K = \{h \cdot k = h \in H, k \in K\} = H = KH.$$

10. If H and K are two subgroups of a group G such that $H \cdot K = K \cdot H$, then prove that $H \cdot K$ is a subgroup of G. (This is the smallest group which contains $H \cup K$).

11. If H and K are two subgroups of a group G, whose orders are relatively prime then $H \cap K = \{e\}$.

12. Let S be a semi-group of finite order. Then prove that S forms a group.

13. Find all the subgroups of the Klein 4-group $V_4 = \{\langle a,b\rangle : a^2 = b^2 = (ab)^2 = e\}$. Is $V_4$ abelian?

14. In a group G, if, for each $a$, b, $(ab)^2 = a^2b^2$, then prove that G is an abelian group.

**15.** In a group G, if, for each $a$, b, $(ab)^{-1} = a^{-1}b^{-1}$, then prove that G is an abelian group.

**16.** Let $\theta : G \longrightarrow G'$ be a group epimorphism. Then, prove that, for any subgroup K of G', $\theta^{-1}(K)$ is a subgroup of G which contains the Kernel of $\theta$.

**17.** If G is a group of order 49 then prove that there is no subgroup of order 2,3,5 or 11 in G.

**18.** Let G be an abelian group. Prove that the subsets

(i) $G^n = \{g^n : g \in G\}$

and (ii) $G_n = \{g \in G : g^n = e\}$.

form subgroups of G for each positive integer n.

**19.** Let G be a group and x be an element of odd order in G. Then there exists an element y in G such that $y^2 = x$.

**20.** Let H and K be two subgroups of a finite group G. Then,

$$\text{for } g \in G,\ g(H \cap K) = gH \cap gK.$$

**21.** Prove that every group of prime order p is always cyclic.

## 2.6 PERMUTATION GROUPS

### 2.6.1 INTRODUCTION

The students are already familiar with permutations as arrangements of the objects in definite order. Initially, the concept of permutation is used to count the number of ways an action or a series of actions can be performed. In this context, permutations are understood as mappings over (or on) a given set of objects which are one-to-one and onto both. The binary operation of their composition (multiplication) is already introduced in chapter one of this book. Keeping in view the importance of permutations, as a group structure, in the further study of theory of groups, a detailed elementary knowledge of the permutations is gathered together in this part of the book.

### 2.6.2 DEFINITION

Let $X = \{a_1, a_2, \ldots, a_n\}$ be a finite set of n objects. A mapping p on X which is one-to-one and onto X (or a bijection) is called a permutation of X. i.e.,

A bijection mapping

$$p : X \longrightarrow X,$$

over X is called a permutation of X.

### 2.6.3 CONSEQUENCES FROM THE DEFINITION

1. Every permutation over X has its domain and range X.
2. Each permutation over X fixes an element of X or transforms it to a different element of X.
3. Each permutation relabels the elements of X in same or in different order.
4. Total number of permutations of a set consisting of n objects is $\lfloor n$ . If X is infinite set then there are infinitely many permutations on X.

**Notation**

A permutation p over the set $X = \{a_1, a_2, \ldots, a_n\}$ is denoted by $p = \begin{pmatrix} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_n \end{pmatrix}$, where $b_i = p(a_i) \in X$, for each $i$.

The upper horizontal row indicates the domain of p and the lower row is formed by the respective images of the domain elements of p. In short $p = \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix}$.

## 2.6.4 RELATED CONCEPTS

## 2.6.5 DEFINITION (EQUALITY)

Let p and q be two permutations on a set $X = \{a_1, a_2, \ldots, a_n\}$.

The permutation p is said to be equal to q, if $p(a_i) = q(a_i)$ for each element $a_i$, $\{1 \le i \le n\}$ in X.

## 2.6.6 MULTIPLICATION (COMPOSITION) OF PERMUTATIONS

If $p = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ p(a_1), & p(a_2), & \ldots, & p(a_n) \end{pmatrix}$ and $q = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ q(a_1), & q(a_2), & \ldots, & q(a_n) \end{pmatrix}$

be two permutations on set $X = \{a_1, a_2, \ldots, a_n\}$,

then the composition of p with q, denoted by

$$p \cdot q = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ p(a_1), & p(a_2), & \ldots, & p(a_n) \end{pmatrix} \cdot \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ q(a_1), & q(a_2), & \ldots, & q(a_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ p(a_1), & p(a_2), & \ldots, & p(a_n) \end{pmatrix} \cdot \begin{pmatrix} p(a_1), & p(a_2), & \ldots, & p(a_n) \\ q(p(a_1)), & q(p(a_2)), & \ldots, & q(p(a_n)) \end{pmatrix}$$

$$= \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ q(p(a_1)), & q(p(a_2)), & \ldots, & q(p(a_n)) \end{pmatrix} = \begin{pmatrix} a_i \\ q(p(a_i)) \end{pmatrix}$$

or $p \cdot q\ (a_i) = q(p(a_i))$, for each $a_i \in X$.

## 2.6.7 REMARKS

1. In the product $p \cdot q$, the action of p is observed first and then that of q (on the images of p) using the left to right rule.
2. Some authors use the notation $p \cdot q$ in which the action of q is observed first and then of p, by

   $$p \cdot q\ (a_i) = p(q(a_i))$$

   following the right to left rule of multiplication.
3. Structure wise the use of notation $p \cdot q$ either way makes no difference but image wise.

   $p(q(a_i)) \neq q(p(a_i))$, for all $i$, if $|X| \geq 3$.

### 2.6.8 DEFINITION (IDENTITY)

A mapping p over X is called identity permutation (mapping) if $p(a_i) = a_i$, for each $a_i \in X$. It is denoted by

$$I = \begin{pmatrix} a_i \\ I(a_i) \end{pmatrix} = \begin{pmatrix} a_i \\ a_i \end{pmatrix}$$

For example, $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is the identity permutation on $X = \{1, 2, 3\}$.

### 2.6.9 DEFINITION (INVERSE)

A permutation q over a set X is called the inverse of permutation $p = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix}$, if

$$q = \begin{pmatrix} p(a_i) \\ a_i \end{pmatrix} = p^{-1}, \text{ for each } a_i \in X.$$

We observe in this context that

$$p \bullet q^{-1} = I = p^{-1} \bullet p$$

For example, if $p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ then $p^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

### 2.6.10 EXAMPLE

If $X = \{1, 2, 3, 4\}$ and $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, then find *(i)* $p \bullet q^2$ *(ii)* $(q \bullet p)^{-1}$.

**SOLUTION**

*(i)* Since $q^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$,

therefore $p \bullet q^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$.

*(ii)* $q \bullet p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$,

where $(q \bullet p)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

The student is advised to verify that

$$(q \bullet p)^{-1} = p^{-1} \bullet q^{-1}$$

## 2.6.11 DEFINITION (CYCLE OR A CYCLIC PERMUTATION)

Let $\delta$ be a permutation on a set $X = \{a_1, a_2, \ldots, a_n\}$, such that $\delta(a_1) = a_2$, $\delta(a_2) = a_3$ and so on $\delta(a_l) = a_1$ for $l \leq n$ and $\delta(a_{l+1}) = a_{l+1}$. Then $\delta$ is called a cyclic permutation or a cycle of length $l$.

It is written as, $\delta = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{l-1} & a_l & a_{l+1} & \ldots & a_n \\ a_2 & a_3 & a_4 & \ldots & a_l & a_1 & a_{l+1} & \ldots & a_n \end{pmatrix}$ which is written in short by $\delta = (a_1\, a_2 \ldots a_l)$.

$\delta$ will be of maximum length n, if $\delta = (a_1\, a_2\, a_3 \ldots a_n)$.

## 2.6.12 CONSEQUENCES FROM THE DEFINITION

1. The number of symbols involved in a cyclic permutation is called its length.
2. The identity permutation is cyclic of length one and denoted by $I = (1)(2)(3) \ldots (n)$.
3. Product of two cycles may or may not be a cyclic permutation.

   for example, $(1\,2\,4\,5) \bullet (3\,4\,5) = (1\,2\,5)(3\,4)$

   and $(1\,2) \bullet (1\,3) = (1\,2\,3)$
4. If there is one element common in the domain of two cycles then their product is a cycle.
5. If the symbols involved in any two cycles form disjoint sets then the cycles are called disjoint cycles.
6. If p is a cycle then Dom (p) = Range (p).

## 2.6.13 THEOREM

Each permutation on a set $X$ ($|X| = n$) is either a cycle or a product of different disjoint cycles.

**Proof**

We know that there is a permutation p of maximum length n showing action on $X = \{a_1, a_2, \ldots, a_n\}$ by

$$p(a_1) = a_2,\quad p(a_2) = a_3 \quad \ldots \quad p(a_n) = a_1$$

All the permutations are not of maximum cyclic length n and there exists permutation q such that

$$q(a_1) = a_2, \ldots, q(a_l) = a_l \quad (l < n)$$

and $q(a_{l+1}) = a_{l+2} \quad \ldots \quad q(a_n) = a_{l+1}$

where $q = (a_1\, a_2 \ldots a_l)(a_{l+1}\, a_{l+2} \ldots a_n)$ is a product of two cycles.

Similarly q may be the product of more than two cycles by

$q = (a_1\ a_2\ \ldots\ a_l)\ (a_{l+1}\ a_{l+2}\ \ldots\ a_{l+k} = a_m)\ (a_{m+1}\ a_{m+2}\ \ldots\ a_{m+w} = a_n)$

where $l + k + w = n$

and q is product of three cycles.

Similarly q could be a product of more than three cycles.

Hence we conclude that each permutation q is itself cyclic or it is a product of two more than two disjoint cycles.

### 2.6.14 THEOREM

All the disjoint cycles on a set X commute (functionally) with each other.

**Proof**

Let $X = \{a_1, a_2, \ldots, a_n\}$. If p and q are any two disjoint cycles of X, then Dom (p) = $X_1$ ($\subseteq$ X) and Dom (q) = $X_2$ ($\subseteq$ X) such that $X_1 \cap X_2 = \phi$ and $X_1 \cup X_2 = X$.

If $a_i \in X$, then p.q $(a_i)$ = q(p($a_i$)), by definition.

If $x \in X$, then $x \in$ Dom (p) or $x \in$ Dom (q) or $x \in X - (\text{Dom } p \cup \text{Dom } q)$.

There are three cases to be discussed separately.

*(i)* If $x \in X - (X_1 \cup X_2)$, then p(x) = x = q(x).

and p.q (x) = q(p(x)) = q(x) = x = q.p(x), which shows that p.q = q.p.

(ii) If $x \in$ Dom (p), then q(x) = x since p and q are disjoint cycles. Also

p.q(x) = q (p(x)) = p(x)

and q.p(x) = p(q(x)) = p(x),

which shows the equality of p.q and q.p.

*(iii)* If $x \in$ Dom (q), then p(x) = x. Also

p . q(x) = q(p(x)) = q(x)

and q . p(x) = p(q(x)) = q(x), showing p.q = q.p.

Hence p.q = q.p in general, which concludes that any two disjoint cycles of X commute together.

**For example**, if p = (2 3 4) and q = (7 8 9) on

X = {1, 2 ,3, 4, 5, 6, 7, 8, 9}

then p . q = (2 3 4) (7 8 9) = (7 8 9) (2 3 4) = q . p

### 2.6.15 DEFINITION (TRANSPOSITION)

A permutation of a cyclic length 2 is called a transposition.

For example, (1 2), (1 3), (2 3) are all the transpositions on the set

$$X = \{1, 2, 3\}.$$

### 2.6.16 THEOREM

Every cyclic permutation on set $X = \{a_1, a_2, \ldots, a_n\}$ is a product of transpositions.

**Proof**

If a permutation $p = (a_1\ a_2\ a_3)$ is a cyclic of length 3.

Then $(a_1\ a_2\ a_3) = (a_1\ a_2)\ (a_1\ a_3)$.

By applying the principle of mathematical inductions on n, we suppose that the result holds for a permutation $p = (a_1\ a_2\ a_3 \ldots\ a_{n-1})$ of cyclic length $(n - 1)$. Thus

$$p = (a_1\ a_2)\ (a_1\ a_3) \ldots\ (a_1\ a_{n-1})$$

Multiplying by $(a_1\ a_n)$, we get that

$$q = p\ .\ (a_1\ a_n) = (a_1\ a_2)\ (a_1\ a_3) \ldots\ (a_1\ a_{n-1})\ (a_1\ a_n)$$

$$= (a_1\ a_2 \ldots\ a_{n-1}\ a_n)$$

is cyclic of length n, which is the product of transpositions. So it holds for cyclic permutations of any finite length.

**Remarks**

Since every permutation on X can be expressed as product of cyclic permutations (disjoint) and every cyclic permutation is product of transpositions, therefore each permutation can be expressed as a product of transpositions.

### 2.6.17 DEFINITION

If a permutation p is a product of odd number of transpositions on a set X, then p is called an odd permutation. Otherwise p is called an even permutation on X.

**For example,** $q = (1\ 2\ 3\ 4) = (1\ 2)\ (1\ 3)\ (1\ 4)$

is an odd permutation of $X = \{1, 2, 3, 4\}$.

and $(1\ 2\ 3) = (1\ 2)\ (1\ 3)$ is an even permutation.

## 2.6.18 REMARKS

1. The identical permutation is considered an even permutation.
2. Every permutation over a set X is either even or odd permutation.
3. Every cyclic permutation of odd length is even and even cyclic permutation of even length is odd.

## 2.6.19 DEFINITION (ORDER OF CYCLIC PERMUTATIONS)

Let $X = \{a_1, a_2, \ldots, a_n\}$ be a finite set of n elements and a cyclic permutation $(a_1, a_2, \ldots, a_n)$ over X will be of maximum length n. If $p = (a_1\ a_2 \ldots a_l)$ of length $l < n$, then of course,

$$p = (a_1\ a_2 \ldots a_l) = (a_2\ a_3 \ldots a_l\ a_1) = (a_3\ a_4 \ldots a_l\ a_1\ a_2)$$

$$= \ldots = (a_l\ a_1\ a_2 \ldots a_{l-1}) = \begin{pmatrix} a_i \\ a_{i+1} \end{pmatrix} \text{ module } l.$$

Keeping in view the composition of permutations, the powers of p, showing actions on $a_i$ . $(1 \le i \le l)$ under the rule.

$$p^k = \begin{pmatrix} a_i \\ a_{i+k} \end{pmatrix} \text{ (modulo } l) \qquad 1 \le i \le l$$

or $p^k(a_i) = a_{i+k}$ (modulo $l$) for each i.

Since $p^l(a_i) = a_{i+l} = a_i$ (modulo $l$) for each i.

Therefore it produces the identity permutation $\begin{pmatrix} a_i \\ a_i \end{pmatrix} = p^l$. This way $l$ is the order of the cyclic permutation p.

Hence, we observe that,

1. The length of a cyclic permutation p is equal to the order of p.
2. Every transposition is of order 2.
3. All the powers of index less than the length of a cyclic permutations are all distinct permutations from each other.
4. If $p = (a_1\ a_2 \ldots a_l)$ then

   $p^{-1} = (a_l\ a_1\ a_{l-1} \ldots a_2) = (a_l\ a_{l-1} \ldots a_1)$,

   where $p \cdot p^{-1} = \text{Identity} = p^{-1} \cdot p$.
5. $p^{-1} = p^{l-x}$ is the cyclic permutation of length same as that of p and hence of same order as that of p.

   or $p^k \cdot p^{l-k}(a_i) = p^{l-k}(p^k(a_i)) = p^{l-k}(a_{i+k}) = a_i = a_{i+k+l-k}$

6. If $l$ is even then $p^2$ is product of two cyclic purmutations of length $\frac{l}{2}$ each.

7. If p is of composite length $l$, then all the powers of p are not cycles of sme length $l$.

8. All the powers of n cyclic permutation p of a cyclic length are of same length $l$, if and only if $l$ is a prime length of p.

**For example,** If $p = (a_1\ a_2\ a_3\ a_4\ a_5)$,

then $p^2 = (a_1\ a_3\ a_5\ a_2\ a_4)$,

$p^3 = (a_1\ a_4\ a_2\ a_5\ a_3\ a_1)$,

$p^4 = (a_1\ a_5\ a_4\ a_3\ a_2)$

and $p^5 = (a_1)\ (a_2)\ (a_3)\ (a_4)\ (a_5)$

when $p \cdot p^4 = I = p^4 \cdot p$

If $p = (a_1\ a_2\ a_3\ a_4)$,

then $p^2 = (a_1\ a_3)\ (a_2\ a_4)$.

### 2.6.20 THEOREM

All the powers of a cyclic permutation over a set $X = \{a_1, a_2, \ldots\ldots, a_n\}$ of prime length $l$ form a group of order $l$.

**Proof**

Let $p = (a_1\ a_2 \ldots\ldots\ a_l)$ be a cyclic permutation of prime length $l$. Then distinct powers of p are all of length $l$, where $p^l$ is the identity permutation.

Since $p^k \cdot p^r\ (a_i) = p^r \cdot (p^k(a_i)) = p^r(a_{i+k}) = a_{(i+k)+r} = a_{i+(k+r)}$

$= p^{k+r}\ (a_i)$ (modulo $l$)

i.e., $p^k \cdot p^r = p^{k+r}$

If $(p^k)^l = I = (p^r)^l$ then $(p^{k+r})^l = I$

Their composition as mappings is associative.

Further $(p^k)^{-1}\ (k \le l) = p^{l-k}$ is a cyclic of same length $l$.

where $p^k \cdot p^{l-k}(a_i) = p^k(p^{l-k}(a_i)) = p^k(a_{i+l_k}) = a_{i+l_k+k} = a_{l+i} = a_i$

giving $p^{l-k}$ as inverse of $p^k$.

and $(p^{l-k})^l(a_i) = p^{l(l-k)}(a_i) = a_i$

Giving $p^{l-k}$ of length $l$.

Hence all the powers of p form a group of order $l$ under multiplication (composition).

### 2.6.21 OBSERVATIONS

*(i)* The group of formed above is infact abelian, because

$$p^k.p^r = p^{k+r} = p^{r+k} = p^r.p^k$$

*(ii)* The above formed group is cyclic because, all the elements of this group are powers of the single element p, which are $\{p, p^2, p^3, \ldots\ldots, p^l = I\}$.

*(iii)* If $l$ is not prime, ever then the group generated by p of length $l$ is of order $l$ but all the permutations are not of same length $l$.

### 2.6.22 THEOREM

Let $X = \{a_1, a_2, \ldots\ldots, a_n\}$ be a set n symbols. The set $P_n$, consisting of all the permutations over X forms a group.

**Proof**

Let $p = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix}$ and $q = \begin{pmatrix} a_i \\ q(a_i) \end{pmatrix}$ be two permutations. Then their multiplication (composition)

$$p \cdot q = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix} \cdot \begin{pmatrix} a_i \\ q(a_i) \end{pmatrix} = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix} \cdot \begin{pmatrix} p(a_i) \\ q(p(a_i)) \end{pmatrix}$$

$$= \begin{pmatrix} a_i \\ q(p(a_i)) \end{pmatrix} = \begin{pmatrix} a_i \\ p \cdot q(a_i) \end{pmatrix} \text{ is an other permutation of X.}$$

It thus shows the closure property of composition of permutations.

If $l = \begin{pmatrix} a_i \\ l(a_i) \end{pmatrix}$ is an other permutation not essentially distinct, then

$$(p \cdot q) \cdot l = \begin{pmatrix} a_i \\ l(p.q(a_i)) \end{pmatrix} = \begin{pmatrix} a_i \\ l(q(p(a_i))) \end{pmatrix}$$

$$= \begin{pmatrix} a_i \\ q.l(p(a_i)) \end{pmatrix} = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix} \cdot \begin{pmatrix} p(a_i) \\ q.l(p(a_i)) \end{pmatrix}$$

$$= \begin{pmatrix} a_i \\ p.(q.l)a_i \end{pmatrix} = p \cdot (q \cdot l)$$

for all p, q, $l$ permutations of X and $a_i \in X$.

Hence the associative property of composition is established.

The identity permutation $I = \begin{pmatrix} a_i \\ a_i \end{pmatrix}$, where

$$p \,.\, I\,(a_i) = I(p(a_i)) = p(a_i) = p(I(a_i)) = p.I(a_i) \text{ or all i,}$$

Hence $p \,.\, I = p = I \,.\, p$, for each permutation p on X.

Further, for every permutation $p = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix}$, there exists its inverse permutation

$p^{-1} = \begin{pmatrix} p(a_i) \\ a_i \end{pmatrix}$ or $\begin{pmatrix} a_i \\ p^{-1}(a_i) \end{pmatrix}$,

where $p \,.\, p^{-1} = \begin{pmatrix} a_i \\ p(a_i) \end{pmatrix} \cdot \begin{pmatrix} p(a_i) \\ a_i \end{pmatrix} = \begin{pmatrix} a_i \\ a_i \end{pmatrix} = I = p^{-1} \,.\, p$

Thus the set $P_n$ of all permutations of a set X of n objects forms a group under the operation of their functional composition. It is denoted by $S_n$ and called the symmetric group of degree n.

### 2.6.23 OBSERVATIONS

In $S_n$, all the even permutations form a subset E of $S_n$ and O, the subset of all odd permutations. If the identity permutation is considered as even permutation, then E is a subgroup of $S_n$. The product of odd permutation is not generally an odd permutation, therefore odd permutations, structurewise, are not of much importance.

### 2.6.24 THEOREM

There are half of the even permutations in $S_n$ and half of them are odd.

**Proof**

If E stands for a set of all even permutations and O for odd permutations in $S_n$, then their multiplication is defined by the Cayley's table,

| · | E | O |
|---|---|---|
| E | E | O |
| O | O | E |

The {E, O} forms a group of order 2 under the multiplication defined above.

Let $\mu : S_n \longrightarrow \{1, -1\}$ be a map, defined by

$$\mu(p) = 1, \quad \text{if p is even}$$

$$\mu(p) = -1, \quad \text{if p is odd.}$$

Then M is an epimorphism with Ker $\mu$ = E.

$E = \text{Ker}\ \mu$ being normal subgroup of $S_n$.

By the fundamental theorem on homomorphism it produces $\frac{S_n}{E} \cong C_2$.

Since each coset of $S_n$ by E has equal number of elements, therefore, there are half of the even and half of the odd permutations of $S_n$.

$$\text{or} \qquad \left| \frac{S_n}{E} \right| = |C_2| = 2$$

$$\Rightarrow \qquad \frac{|S_n|}{|E|} = 2$$

$$\text{or} \qquad |E| = \frac{|S_n|}{2}$$

The normal subgroup E consisting of all the even permutations is known as an **Alternating subgroup** $A_n$ of $S_n$, where

$$|A_n| = \frac{|S_n|}{2}$$

The important facts known about $A_n$, $S_n$ are that,

*(i)* No $S_n$ is simple except $n \leq 2$.

*(ii)* All $A_n$ are simple for $n \geq 5$.

*(iii)* $[S_n : A_n] = 2$, except $n = 6$.

### 2.6.25 CONSEQUENCES

*(i)* If $S_n$ is the symmetric group of degree n over $X = \{1, 2, 3, \ldots, n\}$ and $Y \subset X$. Such that $|Y| = m < n$, then $S_m \subset S_n$.

*(ii)* For every $m < n$, then exists a subgroup $S_m$ of $S_n$ (i.e., $S_n$ consists of permutations of a subset Y of X. of order m, leaving rest of the elements of $X - Y$ invariant).

*(iii)* $S_m \cong S_n$ iff $m = n$.

*(iv)* If $X = \{1, 2, 3\}$ and $p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

then $p \cdot q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

and $q \cdot p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \neq p \cdot q$

Thus $S_n$ is a non-abelian group.

Since $S_3 \leq S_n$ $(n \geq 3)$, therefore all supper groups of $S_3$ are non-abelian.

*Note.* We know that $S_n$ is abelian if $n \leq 2$ and groups of order 4 and 5 are also abelian groups, therefore $S_3$ is the smallest non-abelian group of order 6.

## EXERCISE 2.4

1. Mark each of the following statement True (T) or False (F).

   *(i)* A permutation is a bijective function.

   *(ii)* The symmetric group $S_{29}$ has 29 elements.

   *(iii)* The symmetric group $S_3$ is cyclic.

   *(iv)* Every permutation is a cycle.

   *(v)* Every cycle is a permutation.

   *(vi)* The number of odd permutations in $S_n$ is equal to the number of even permutations in $S_n$.

   *(vii)* $S_n$ is not cyclic for $n \geq 3$.

   *(viii)* Every transposition is an odd permutation.

   *(ix)* The product of two cycles may be a cycle.

   *(x)* $S_4$ is the smallest non-abelian group.

2. Let $X = \{1, 2, 3, 4, 5, 6\}$ be a set and

   $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$, $r = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$ be three permutations on X.

   Compute *(a)* $p^2q$ *(b)* $rq^2$ *(c)* $p^{-1} \cdot q \cdot r$ *(d)* $(pq)^{-1}$ *(e)* $(pqr)^{-1}$

3. Find the products and express them as products of cyclic permutations on mutually disjoint sets.

   *(a)* (1 2 3 4) (5 2 3 4 1)

   *(b)* (1 4) (2 3 5) (3 5) (4 5)

   *(e)* (1 2 4) (1 3 6 5 4)

   *(f)* (1 2 3 4 5) (2 5 4 3 1) (1 5 3 2 4)

4. Find all the subgroups of the group $S_3$, $S_4$ and $S_5$.

5. Determine whether the given permutations are odd or even on
$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 3 & 6 & 1 & 7 & 9 & 8 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$

6. Find the order of each of the given permutations

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 5 & 3 & 1 & 8 & 6 & 7 & 10 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} a & b & c & d & e & f & g & h \\ h & a & f & c & d & e & h & g \end{pmatrix}$

7. Let $X = \{1, 2, 3, 4, 5\}$ be a set and $p = (1\ 2\ 3\ 4)$, $q = (2\ 3\ 5)$ be two permutations. Then find;

(i) $p^2$, $q^2$, $p \cdot q$ and $q \cdot p$.

(ii) The orders of p and $q \cdot p$.

(iii) $p^{-1}$ and $(pq)^{-1} \neq p$

(iv) The length of $q \cdot p^2$.

(v) The group generated by $(1\ 2\ 3\ 4)$.

(vi) Whether $p^2 \cdot q$ is odd or even permulation.

(vii) Whether $q^2$ is odd or even.

8. Make the following products and examine;

(i) $(1\ 2\ 3\ 4\ 5)(1\ 2\ 3)$

(ii) $(1\ 2)(1\ 3)(1\ 4)$

(iii) $(1\ 2)(3\ 4\ 5$.

# 3

# VECTOR SPACES AND THEIR LINEAR TRANSFORMATIONS

## 3.1 INTRODUCTION

In Chapter 2, we have discussed the algebraic structures of groups and semi-groups. They are one-operational structures. In this chapter, we characterize some more complex algebraic systems, called Rings, Fields and Vector spaces. These all are two-operational systems and are groups or semi-groups under one of the two operations. The important thing here to note is, that the number systems of natural numbers, integers, rationals, reals and complex numbers are used as a ground to generalize their properties in the formation of Rings, Fields and Vector Spaces.

For the sake of revision, elementary knowledge of the real numbers with the properties of their usual operations of addition (+) and multiplication(·) is given below:-

### 3.1.1 REAL NUMBER SYSTEM

#### $R_1$ : ADDITIVE PROPERTIES

(a) For every $m, n \in R$, $m + n = k \in R$ (closure)

(b) For every $m, n, r \in R$, $(m + n) + r = m + (n + r)$ (associative)

(c) For real number 0, $m + 0 = m = 0 + m$, $\forall m \in R$ (identity)

(d) For every real number m, there is a real number $-m$ such that

$m + (-m) = 0 = (-m) + m$ (inverse)

(e) For every $m, n \in R$, $m + n = n + m$ (commutative)

## $R_2$ : MULTIPLICATIVE PROPERTIES

(f) For every $m, n \in R$, $m \cdot n = t \in R$ (closure)

(g) For every $m, n, k \in R$, $(m \cdot n) \cdot k = m \cdot (n \cdot k)$ (associative)

(h) For real number 1 and $m \in R$, $1 \cdot m = m = m \cdot 1$ (identity)

(i) For every non-zero real $n \in R$, $1/n = n^{-1} \in R$, such that
$n \cdot n^{-1} = 1 = n^{-1} \cdot n$ (inverse)

(j) For every $m, n \in R$, $m \cdot n = n \cdot m$ (commutative)

## $R_3$ : DISTRIBUTIVE PROPERTIES

The operation of multiplication is distributive over addition.

For every $m, n, r \in R$,

(k) $m \cdot (n + r) = m \cdot n + m \cdot r$ (right distribution)

(l) $(m + n) \cdot r = m \cdot r + n \cdot r.$ (left distribution)

**NOTE** Since $m + (n \cdot r) \neq (m + n) \cdot (m + r)$, therefore, addition is not distributive over multiplication.

Instead of considering this particular set R, if any set F is considered with general binary operations of * and o, we define the algebraic structures, in general, of rings and fields.

### 3.1.2 DEFINITION

The system $(F, *, \circ)$ is called Ring if ;

(i) $(F, *)$ is a commutative group

(ii) $(F, \circ)$ is a semi-group

and (iii) The binary operation $\circ$ over $*$ is distributive (left and right),

$\forall a, b, c \in F$ $\quad a \circ (b * c) = (a \circ b) * (a \circ c),$

$(a * b) \circ c = (a \circ c) * (b \circ c),$

**NOTE** If $*$ and o correspond to the binary operations of addition (+) and multiplication (·) of real numbers respectively then (R, +, ·) forms a ring.

### 3.1.3 DEFINITION

The system (F, *,o) is called field if;

(i) (F, $*$) is a commutative group.

(ii) ($F^*$, o) is a commutative group, when $F^*$ is a subset of F excluding $*$ – identity of F.

and (iii) The binary operation o over $*$ is distributive.

For every $a$, b, c $\in$ F,

$a \circ (b * c) = (a \circ b) * (a \circ c)$ (right distributive)

$(a * b) \circ c = (a \circ c) * (b \circ c)$ (left distributive)

**NOTE 1** If $*$ and o correspond to the binary operations of addition (+) and multiplication (·) of real numbers respectively then (R, +, ·) forms a field.

**NOTE 2** Every field is a ring but the converse does not hold in general.

### 3.1.4 DEFINITIONS

(i) A ring (F, $*$, o) is a commutative ring if set F is o-commutative.

(ii) A ring (F, $*$, o) is a ring with identity if set F contains o-identity.

(iii) A ring (F, $*$, o) is a commutative ring with identity if F is o-commutative and contains o-identity.

(iv) A ring (F, $*$, o) forms a field if it is a commutative ring with identity and ($F^*$, o) forms an abelian group where

$$F^* = F - \{* - \text{identity}\}$$

(v) If ($F^*$, o) forms a non-abelian group then it is called a skew field or a division ring.

Thus every commutative skew field forms a field structure.

**NOTE** Although the binary operations $*$ and o correspond to the objects contained in F but we, for convenience, shall denote them '+' and '·' respectively unless otherwise mentioned.

### 3.1.5 EXAMPLES

1. $(Q, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ form fields

2. $(Z, +, \cdot)$ is a commutative ring with identity but not a field.

3. The set $\{0\}$, consisting of zero alone, forms a ring under the binary operations of addition and multiplication of numbers. It is called null-ring, which is the smallest ring structure. It is not a field.

4. If the set E consists of all even integers together with 0 then $(E, +, \cdot)$ is a commutative ring without identity.

5. If set F[x] consists of all polynomials with coefficients from F involving variable x then $(F[x], +, \cdot)$ forms a ring with identity but not a field. Why?

6. If set $R_m$ is of all residue classes of integers modulo m then $(R_m, +, \cdot)$ forms a commutative ring with identity when,

$$[m] + [n] = [m + n] \quad \text{(modulo m)}$$

and $$[m] \cdot [n] = [m \cdot n] \quad \text{(modulo m)}$$

If m is a prime number then $R_m$ forms a field. Verify.

### 3.1.6 DEDUCTIONS FROM THE DEFINITIONS

#### 1. CANCELLATION LAWS

Let $(R, +, \cdot)$ be a ring and $m, n, r \in R$ such that

$$m + r = n + r$$

then $$m = n \quad \text{(right cancellation)}$$

Since $m + r = n + r$, therefore,

$$(m + r) - r = (n + r) - r \quad \text{(additive inverse)}$$

$$\Rightarrow \quad m + (r - r) = n + (r - r) \quad \text{(associative property)}$$

$$\Rightarrow \quad m + 0 = n + 0 \quad \text{(inverse property)}$$

$$\Rightarrow \quad m = n \quad \text{(additive identity)}$$

Similarly, the left cancellation law follows.

i.e. $r + m = r + n \rightarrow m = n$

2. Let $(R, +, \cdot)$ be a ring and $x \in R$. Then

$$x + x = x \text{ if and only if } x = 0$$

If $x = 0$ then $0 + 0 = 0$, by definition.

Conversely, if $x + x = x = x + 0$

then $x = 0$, by the left cancellation law.

3. If $(R, +, \cdot)$ is a ring and 0 is the additive identity of the ring,

then $r \cdot 0 = 0 = 0 \cdot r, \quad \forall\, r \in R.$

We know that $r \cdot 0 = r \cdot (0 + 0)$ (identity property)

$= r \cdot 0 + r \cdot 0$ (distributive property)

$\Rightarrow r \cdot 0 = 0$

Similarly, $0 \cdot r = (0 + 0) \cdot r$

$= 0 \cdot r + 0 \cdot r$

$\Rightarrow 0 \cdot r = 0$

### 3.1.7 THEOREM

Let $(R, +, \cdot)$ be a ring. Then, for all $a, b \in R$,

(i) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

and (ii) $(-a) \cdot (-b) = ab$

**PROOF** (i) Take $a \cdot b + (-a) \cdot b = (a + (-a)) \cdot b$ (distributive property)

$= 0 \cdot b$ (inverse property)

$= 0$, by 3.1.6 (3),

which shows that $(-a) \cdot b$ is the additive inverse of $a \cdot b$.

Hence $(-a) \cdot b = -(a \cdot b)$

Similarly, $a \cdot (-b) + a \cdot b = a \cdot (-b + b)$

$= a \cdot 0$

$$= 0$$

Hence $a \cdot (-b) = -(a \cdot b)$

Consequently $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

(ii) $(-a) \cdot (-b) = -((a) \cdot (-b)) = -(-(a \cdot b))$

$$= a \cdot b,$$

which proves (ii), for all $a, b \in R$.

### 3.1.8 THEOREM

If a ring $(R, +, \cdot)$ is not a null-ring and contains the multiplicative identity e then $e \neq 0$, the additive identity of R.

**PROOF** If $e = 0$ and R is not a null-ring then, for every element $a \in R$,

$a \cdot e = 0$ by 3.1.6 (3)

$= a$, as e is the multiplicative identity of R,

which implies that $a = 0$

$\Rightarrow$ R is a null-ring.

This is contrary to our assumption that R is not a null-ring.

Hence the assertion follows.

**COR.** Any field F contains at least two elements, which are identities under both the operations on F.

## EXERCISE 3.1

**1.** Which of the following systems are rings? For those which are rings, classify each according to the definitions 1 through 4.

(i) $(S, +, \cdot) : S = \{a + b\sqrt{2}, a, b \in Z\}$.

(ii) $(T, +, \cdot) : T = \{(x + y\sqrt[3]{2}, x, y \in Z\}$.

(iii) $(E, +, \cdot) : E = \{2n, n \in N\}$

(iv) $(Z_p, +, \cdot)$ : $Z_p$ is the set of residue classes mod p (p is prime number)

(v) $(Z_7, +, \cdot)$ : $Z_7$ is the set of integers mod 7.

(vi) $(F[x], +, \cdot)$ : F[x] is the set of all polynomials with coefficients from the field F of degree $\geq n$. (n is a +ve integer). '+' and '·' denote the usual addition and multiplication of the polynomials.

(vii) $(F[x], +, \cdot)$ : F[x] is the set of all polynomials with coefficients from the field F of degree $\leq 5$.

(viii) $(M, +, \cdot)$ : M is the set of all matrices of order n x n, with real entries. '+' and '·' denote the usual addition and multiplication of matrices.

(ix) $(G, +, \cdot)$ : $G = \{x + iy : x, y \in Z\}$ is the set of Gaussian integers.

(x) $(A, +, \cdot)$ : Where $A = \left\{p + q\sqrt{2}\ , p, q \in Q\right\}$

(xi) $(\mathbb{C}, +, \cdot)$ : When $\mathbb{C}$ is the set of all complex numbers.

(xii) $(R, +, \cdot)$ : Where R is the set of all real numbers.

(xiii) The set {0, 1, 2, 3}, where the addition and multiplication over the set are defined by the tables as below ;

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| · | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

2. Prove or disprove the following statements;

(i) The set Z of all integers does not form a field under the binary operations of addition and multiplication of numbers.

(ii) $(Z, +, \cdot)$ forms a ring where Z is the set of all integers, '+' is the usual addition and '·' is defined by $a \cdot b = 0, \forall a, b \in Z$.

(iii) $(M, +, \cdot)$ is a commutative ring, when M is the set of all 2 x 2 matrices over the reals.

(iv) In a ring R, the equation $a + x = b$, has a unique solution in R. $a, b \in R$.

(v) $(S, +, \cdot)$ is a ring with identity element, where addition (+) and multiplication (·) over $S = \{a, b\}$ are defined by the following tables;

| + | $a$ | b |
|---|---|---|
| $a$ | $a$ | b |
| b | b | $a$ |

| · | $a$ | b |
|---|---|---|
| $a$ | $a$ | $a$ |
| b | $a$ | b |

(vi) A ring can contain exactly two elements because there are two binary operations involved.

(vii) The system $(E, +, \cdot)$ is a field.

(viii) The system $(R, +, \cdot)$, where '+' is usual addition of real numbers and $m \cdot n = m - n, \forall\ m, n \in R$, forms a division ring.

(ix) Let X be a non-empty set and P(X) be the power set of X. Then $(P(X), \cup, \cap)$ forms a commutative ring.

(x) $(P(X), \Delta, \cap)$ is a commutative ring, where P(X) is the power set of a non-empty set X, $A \Delta B = (A \cup B) - (A \cap B)$ and $\cap$ has its usual meaning.

**3.** Show that the set $\{0, 1, w, w^2\}$ forms a field under the binary operations of addition mod 2 and multiplication of complex numbers, $w = \dfrac{-1 + \sqrt{-3}}{2}$.

**4.** Show that the system (S, +, .) of Q.2 (v) forms a field.

## 3.2 VECTOR SPACE

Vector space is a mathematical system which involves three mathematical objects, of a field, an abelian group and a mapping, called scalar multiplication. Independently, all the three objects are known by the reader from the study of chapter 2 and 3.1 of this chapter. We are going to use them in the formation of a vector space in this section. The author feels that, before properly defining this algebraic structure of a vector space, it is useful to refresh the study of vectors in the Euclidean plane and the properties of their binary operations of addition and multiplication.

### 3.2.1 VECTORS IN $R^2$

From the field R of all real numbers, we construct the Cartesian product of R with R

i.e. $R \times R = \{(x, y) : x, y \in R\} = R^2$

and named every element $(x, y) \in R^2$ as a vector in the plane in chapter 1.

### 3.2.2 BINARY OPERATIONS OF VECTORS IN $R^2$

#### (i) ADDITION OF VECTORS

Let $(x_1, y_1)$ and $(x_2, y_2)$ be two vectors of $R^2$. Then their addition (+) is defined as,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 , y_1 + y_2) \in R^2.$$

The addition of vectors is understood as 'component wise addition' of vectors in the plane.

The associative property of the addition of vectors in $R^2$ is induced from the additive associative property of R.

Zero vector $(0, 0) \in R^2$, such that

$$(0, 0) + (x, y) = (0 + x, 0 + y) = (x + 0, y + 0) = (x, y) + (0, 0)$$
$$= (x, y) = (x, y) + (0, 0)$$

For every vector $(x, y) \in R^2$, $(-x, -y) \in R^2$, such that

$$(x, y) + (-x, -y) = (x + (-x), y + (-y)) = (0, 0)$$
$$= (-x + x ,-y + y)$$
$$= (-x, -y) + (x, y)$$

$(-x, -y)$ is denoted by $-(x, y)$, the additive inverse of $(x, y)$.

It thus proves the **associative, identity** and **inverse properties** of addition of vectors in $R^2$.

Moreover, the commutative additive property of vectors is induced from the additive commutative property in R.

The above properties gathered together form $R^2$ a commutative group under the operation of vector addition.

## (ii) SCALAR MULTIPLICATION OF VECTORS IN $R^2$

For $\alpha \in R$, a mapping $\varphi_\alpha$ associated with $\alpha$, from $(R \times R) \to R \times R$ defined by $\varphi_\alpha (x, y) = \alpha \cdot (x, y) = (\alpha x, \alpha y) \in R^2$, is called, scalar multiplication by $\alpha$ to the vector $(x, y)$.

Such mappings exist, for each $\alpha \in R$.

The following properties of scalar multiplication are observed;

(i) $\forall\ \alpha \in R$, and $(x_1, y_1), (x_2, y_2) \in R^2$

$$\begin{aligned}\alpha \cdot [(x_1, y_1) + (x_2, y_2)] &= \alpha[(x_1 + x_2, y_1 + y_2) \\ &= (\alpha(x_1 + x_2), \alpha(y_1 + y_2)) \\ &= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2) \\ &= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2) \\ &= \alpha(x_1, y_1) + \alpha(x_2, y_2)\end{aligned}$$

Which implies that vector addition is invariant under scalar multiplication.

$$\begin{aligned}\text{(ii)}\ \forall\ \alpha, \beta \in R,\ (\alpha + \beta) \cdot (x, y) &= ((\alpha + \beta)x, (\alpha + \beta)y) \\ &= (\alpha x + \beta x, \alpha y + \beta y) \\ &= (\alpha x, \alpha y) + (\beta x, \beta y) \\ &= \alpha (x, y) + \beta(x, y)\end{aligned}$$

$$\begin{aligned}\text{(iii)}\ \forall\ \alpha, \beta \in R,\ \alpha(\beta(x, y)) &= \alpha(\beta x, \beta y) = (\alpha\beta x, \alpha\beta y) \\ &= \alpha\beta(x, y)\end{aligned}$$

(iv) 1 (as multiplicative identity) $\in R$ and

$$1 \cdot (x, y) = (1x, 1y) = (x, y)\ ,\ \forall (x, y) \in R^2$$

Similarly, we can extend this idea to $R^3$ consisting of triplets

$$(x_1, x_2, x_3) \in R^3\ ,\ \forall\ x_1, x_2, x_3 \in R$$

i.e. $R^3 = R \times R \times R = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in R\}$

Addition defined over these triplets similar to the addition (component wise) in $R^2$ forms $R^3$ an **Abelian group.**

For further extension to the set $R^n$ of all n-tuples of R, similar 'component wise' vector addition and scalar multiplication by the elements of R has the same properties.

We define formally the system of a vector space over a given field structure F.

### 3.2.3 DEFINITION OF A VECTOR SPACE

Let F be a Field. A non-empty set V together with two binary operations (+) and (·) forms the algebraic structure of a vector space our the field F, if;

(a) V forms an additive (+ve) abelian group.

(b) The scalar multiplication (·) as a function from F x V into V observes the following properties;

(i) $\forall\ \alpha \in F, x, y \in V, \quad \alpha\cdot(x + y) = \alpha x + \alpha y$

(ii) $\forall\ \alpha, \beta \in F, \quad (\alpha + \beta)\cdot x = \alpha x + \beta x \quad , \quad \forall\ x \in V.$

(iii) $\forall\ \alpha, \beta \in F, \forall\ x \in V\ ;\ \alpha(\beta x) = (\alpha\beta)\, x$

(iv) For $e_F$, the identity of F, $e_F \cdot x = x \quad , \quad \forall x \in V.$

### NOTATION

We will observe the notation V(F) for a vector space V over the field F. Elements of V are called vectors where that of F are called scalars. Vector spaces over the real and complex fields are called real and complex vector spaces respectively.

**NOTE** The properties enumerated in 3.2.3 (a) and (b) from (i) to (iv) above are infact the properties of spaces of vectors in $R^2, R^3 \cdots, R^n$ etc.

### 3.2.4 OBSERVATIONS

(i) For every vector space, the need for a field is hidden in its structure. We cannot therefore imagine about a vector space without a field.

(ii) Elements of a vector space are constructed from the elements of the field in some sense.

### 3.2.5 EXAMPLES OF VECTOR SPACE

1. Every field is a vector space over itself. Thus R and Q are vector space.

2. The set (F,+,·) of all real valued continuous functions in the interval [0, 1] forms a vector space over the real field R.

where, $\forall\ f, g \in F$ and $m \in [0, 1]$

$$[f + g](m) = f(m) + g(m) \quad \text{(Addition of functions)}$$

$$[(f + g) + h](m) = (f + g)(m) + h(m)$$

$$= [f(m) + g(m)] + h(m)$$

$$= f(m) + [g(m) + h(m)] = [f + (g + h)](m)$$

$$\alpha \cdot f(m) = \alpha(f(m)) \quad \text{(scalar multiplication)}$$

(i) If f and g are continuous then so is f + g.

(ii) Associative property of addition is clearly defined above.

(iii) $(f + 0_f)(m) = f(m) + 0_f(m) = f(m) + 0 = f(m) = (0_f + f)(m)$

$0_f$ is a Zero-function under which image of every member in its domain is zero. Zero-function is the additive identity of the set of all functions.

(iv) $(-f)(m) = -f(m)$, where $(-f)$ is additive inverse of f which is also continuous.

(v) $(f + g)(m) = f(m) + g(m) = g(m) + f(m) = (g + f)(m)$, for all $f, g \in F$

Thus F forms an abelian group under addition of functions.

For the scalar multiplication defined above,

$\alpha \cdot f$ is continuous in [0, 1], for every $\alpha \in R$.

(vi) $(\alpha + \beta)\cdot f = \alpha f + \beta f$ (where $[(\alpha + \beta) f](m) = (\alpha + \beta) f(m)$

$$= \alpha f(m) + \beta f(m)$$

$$= (\alpha f)(m) + (\beta f)(m)$$

and also $$= [\alpha f + \beta f](m)$$

(vii) $\alpha[f + g] = \alpha f + \alpha g$

(viii) $\alpha \cdot (\beta f) = (\alpha\beta) \cdot f$

(ix) $1 \cdot f = f$

Hence continuous functions on the interval [0, 1] form a vector space over the field R of reals.

3. The set D of all derivable functions in the interval [0, 1] forms a vector space over the field R.

It is left as an exercise to verify all the properties of vector space.

4. The set $P_n(x)$ of all polynomials over R in variable x of degree $\leq n$ forms a vector space over R.

If $f_n(x) = a_0 + a_1x + \cdots\cdots\cdots\cdots + a_nx^n$

and $g_n(x) = b_0 + b_1x + \cdots\cdots\cdots\cdots + b_n x^n$, $a_i, b_i \in R$.

Then, $f_n(x) + g_n(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots\cdots\cdots\cdots + (a_n + b_n)x^n$ is a polynomial of $P_n(x)$

The associative additive property is induced from the additive associative property of R.

The zero-polynomial $f_0(x) = 0$ of degree zero acts as the additive identity of $P_n(x)$.

and $-f_n(x) = -a_0 + (-a_1)x + \cdots\cdots + (-a_n)x^n$ is the additive inverse of $f_n(x)$.

Commutative property follows from the commutative property of R. Hence $P_n(x)$ is an additive abelian group.

The scalar multiplication of $\alpha \in R$ by $f_n(x)$ is defined by,

$$\alpha \cdot f_n(x) = \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x_2 \cdots\cdots + (\alpha a_n)x^n \in P_n(x)$$

It observes properties 3.2.3 b(i — iv) of scalar multiplication which can easily be verified. So that $P_n(x)$ forms a vector space over R.

5. Consider the set S of all solutions of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad \cdots\cdots \text{(i)}$$

The solutions obtained are of the type $e^{2t}$, $e^t$ ... etc.

The sum $e^t + e^{2t}$ or $e^{2t} + e^t$ of two solutions is also a solution. Associative property follows from the associative property of $\mathbb{C}$. 0 is a solution of (i), hence $\in$ S. If $e^t$ is a solution then $-e^t$ is also a solution. Commutative property is obvious. So all the solutions of (i) form a commutative group under addition.

For the scalar multiplication, $\alpha, \beta \in \mathbb{C}$

$\alpha \cdot e^t$ is also a solution.

Moreover,

(i) $(\alpha + \beta)\cdot e^t = \alpha e^t + \beta e^t$,

(ii) $\alpha\cdot(e^{t_1} + e^{t_2}) = \alpha e^{t_1} + \alpha e^{t_2}$

(iii) $\alpha\cdot(\beta e^t) = (\alpha\beta)e^t$

(iv) $1\cdot e^t = e^t$.

Hence the solutions of (i) form a vector space over $\mathbb{C}$.

6. The set M of all m x n matrices over the field R, under the binary operation of matrix addition and scalar multiplication, forms a vector space over R.

   The student is advised to verify when the concept of a matrix is introduced with its addition and scalar multiplication in chapter 4.

7. The set $V = \{a + \sqrt{3}\, b : a, b \in R\}$ forms a vector space over the real field R.

8. The set $\mathbb{C} = \{x + iy : i = \sqrt{-1}, x, y \in R\}$ of all complex numbers forms a vector space over the field R of real numbers.

## 9. THE COMPLEX SPACE $\mathbb{C}$

The set $\mathbb{C} = \{(x, y) : x, y \in R\}$ of all complex numbers, forms a vector space under the binary operations of addition of complex numbers and scalar multiplication of complex numbers by real (complex) such that

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$\alpha\cdot(x, y) = (\alpha x, \alpha y), \alpha \in R.$$

## 3.2.6 CONSEQUENCES FROM THE DEFINITION

The assertions proved in the following theorem follow from the definition of a vector space V over a field F.

### 3.2.7 THEOREM

Let V be a vector space over the field F, $0_V$ and $0_F$ be the additive identities of V and F respectively. Then,

(i) $\alpha \cdot 0_V = 0_V$ , $\forall\, \alpha \in F$.

(ii) $0_F \cdot v = 0_V$ , $\forall\, v \in V$,

(iii) $(-\alpha)\cdot v = \alpha(-v) = -(\alpha v)$ , $\forall\, \alpha \in F,\ \forall\, v \in V$

(iv) $\alpha \cdot v = 0_V \Rightarrow$ either $\alpha = 0_F$ or $v = 0_V$, $\forall \alpha \in F, \forall v \in V$.

(v) $\alpha\cdot(v_1 - v_2) = \alpha v - \alpha v_2$ , $\forall\, \alpha \in F,\ \forall\, v_1, v_2 \in V$.

**PROOF**

(i) $\alpha \cdot v \in V$ , by scalar multiplication, $\alpha \in F, v \in V$.

Then, $\alpha \cdot v = \alpha\cdot(v + 0_V)$ , $0_V$ is the additive identity of V,

$= \alpha v + \alpha 0_V$ , distributive property of scalar multiplication.

But, $\alpha v + 0_V = \alpha v$

Therefore, $\alpha v + \alpha 0_V = \alpha v + 0_V$

$\Rightarrow \alpha 0_V = 0_V$ , $\forall\, \alpha \in F$

(ii) $\alpha\cdot v = (\alpha + 0_F)v$ , $0_F$ is the additive identity of F.

$= \alpha v + 0_F v$ (distributive property)

$\alpha v \in V$, $0_F v$ is the additive identity of V,

Thus $0_F v = 0_V$.

(iii) From (ii), $0_V = 0_F\cdot v$ , $\forall\, v \in V$

$= [\alpha + (-\alpha)]\cdot v$ ($0_F$ is the identity of F)

$= \alpha\cdot v + (-\alpha)\cdot v$ (Distributive property)

$\Rightarrow$ $(-\alpha)\, v = -(\alpha v)$ (Inverse property)

Similarly from (i) $0_V = \alpha\cdot 0_V \Rightarrow \alpha\cdot[v + (-v)] = \alpha\cdot v + \alpha\,(-v)$

$$\Rightarrow \alpha(-v) = -(\alpha v).$$

(iv) Suppose $\alpha \cdot v = 0_V$ and $\alpha \neq 0_F$.

Then $\alpha$ has its inverse in F. Therefore,

$$\alpha \cdot v = 0_V$$
$$\Rightarrow \alpha^{-1}(\alpha \cdot v) = \alpha^{-1} \cdot 0_V$$
$$\Rightarrow (\alpha^{-1} \cdot \alpha) \cdot v = 0_V \text{ (from (i))}$$
$$\Rightarrow I_F \cdot v = 0_V \text{ (where } I_F \text{ is the multiplicative identity of F).}$$
$$\Rightarrow v = 0_V \text{ (definition)}$$

Hence $\alpha \cdot v = 0_V \Rightarrow$ either $\alpha = 0_F$ or $v = 0_V$.

**Remark.**

If $v \neq 0_V$ and $\alpha \neq 0_F$, then $\alpha \cdot v \neq 0_V$

Thus $\alpha.v = 0_V$ and $v \neq 0_V$ implies that $\alpha = 0_F$.

(v) L.H.S. $= \alpha \cdot [v_1 - v_2] = \alpha \cdot [v_1 + (-v_2)]$

$= \alpha v_1 + \alpha(-v_2)$

$= \alpha v_1 - \alpha v_2$ (from (iii)).

$=$ R.H.S.

### 3.2.8 CANCELLATION LAWS

Let V(F) be a vector space over the field F. Then the following cancellation laws hold within V(F);

(i) $\alpha \cdot v = \beta \cdot v \Rightarrow \alpha = \beta$, $\forall\, \alpha, \beta \in F$, $\forall\, v \neq 0 \in V$,

(ii) $\alpha \cdot v_1 = \alpha \cdot v_2 \Rightarrow v_1 = v_2$, $\forall\, 0 \neq \alpha \in F$, $\forall v_1, v_2 \in V$.

**PROOF**

(i) $\alpha \cdot v = \beta \cdot v \Rightarrow \alpha v - \beta v = 0_V$

$\Rightarrow \alpha \cdot v + (-\beta)\, v = 0_V$

$\Rightarrow [\alpha + (-\beta)]\, v = 0_V$

$v \neq 0_V \Rightarrow \alpha + (-\beta) = 0_F$ (from 3.2.7 (ii))

$\Rightarrow \alpha = \beta$

(ii) $\alpha \cdot v_1 = \alpha \cdot v_2 \Rightarrow \alpha \cdot v_1 - \alpha v_2 = 0_V \Rightarrow \alpha[v_1 + (-v_2)] = 0_V$

$\alpha \neq 0_F \Rightarrow v_1 + (-v_2) = 0_V$

$\Rightarrow v_1 = v_2$

## EXERCISE 3.2

1. If $P_2(t) = \{a_0 + a_1 t : a_0, a_1 \in R, t \text{ is a real variable}\}$
and $a_0 + a_1 t,\ b_0 + b_1 t \in P_2(t)$ such that
$(a_0 + a_1 t) + (b_0 + b_1 t) = (a_0 + b_0) + (a_1 + b_1)t \in P_2(t)$
and $\alpha \in R,\ \alpha(a_0 + a_1 t) = \alpha a_0 + (\alpha a_1)t \in P_2(t)$
Then show that $P_2(t)$ is a real vector space.

2. If $R^3 = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in R\}$ and the elements of $R^3$ are added component wise, with scalar multiplication defined by,

$$\alpha.(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3) \in R^3,\ \alpha \in R,$$

then show that $R^3$ forms a vector space over R.

3. Let $V_1 = \{(a_1, a_2, a_3, a_4) : a_1, a_2, a_3, a_4 \in R;\ a_1 + a_2 + a_3 + a_4 = 0\}$
Then, with the usual vector addition and scalar multiplication of $R^4$, $V_1$ forms a vector space. Prove or disprove.

4. Let $V = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} : a_1, a_2, a_3 \in R \right\}$ be the set of all 3 x 3 diagonal matrices with entries from the real field R. Then V, together with the usual matrix addition and scalar multiplication of matrices, forms a vector space. Prove or disprove.

5. Let $M_2 = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : d(A) = 0 \right\}$ be set of all 2 x 2 real singular matrices. Then, under the usual matrix addition and scalar multiplication, $M_2$ forms a vector space over R. Prove or disprove.

6. Show that the set of all infinite sequences which converge to zero forms a vector space over the real field R under their usual addition and scalar multiplication.

7. Show that the set of all convergent infinite sequences under their usual addition and scalar multiplication forms a vector space over the real field.

8. Let V be a vector space over F. Prove the following, for all $a, b \in F$ and for all vectors u and $v \in V$;

(i) $-(-u) = u$.

(ii) $-(u + v) = -u - v$

(iii) $-a(-u) = au$

(iv) $(a-b)\cdot u = au - bu$

## 3.3 LINEAR COMBINATIONS OF VECTORS OF V(F)

Let $\upsilon_1, \upsilon_2, \ldots\ldots, \upsilon_n$ be elements of vector space V(F). Any vector $\upsilon \in V(F)$ expressed in the form, $\alpha_1\upsilon_1 + \alpha_2\upsilon_2 + \ldots\ldots + \alpha_n\upsilon_n$, $\alpha_1 \in F$, is said to be a linear combination of the vectors $\upsilon_1, \upsilon_2, \ldots\ldots, \upsilon_n$ over the field F and written as

$$\upsilon = \alpha_1\upsilon_1 + \alpha_2\upsilon_2 + \ldots\ldots + \alpha_n\upsilon_n$$

### 3.3.1 EXAMPLES

1. The vector $(1, 3, 6) \in R^3$ is a linear combination of the vectors

$$\upsilon_1 = (1, 1, 2),\ \upsilon_2 = (2, 1, 3),\ \upsilon_3 = (1, -1, 2)$$

over the rational or the real field R.

i.e. for $\alpha_1 = 8,\ \alpha_2 = -4,\ \alpha_3 = 1$

$$(1, 3, 6) = 8(1, 1, 2) - 4(2, 1, 3) + 1(1, -1, 2)$$

2. Show, whether the vector $(1, 3, -1) \in R^3$ is a linear combination of $(2, 3, -1)$, $(0, -1, 2)$ and $(1, -3, 1)$ over the real field.

Suppose that the vector $(1, 3, -1)$ is the linear combination of the given three vectors, then there exist $\alpha_1, \alpha_2, \alpha_3 \in R$ (field), such that

$$(1, 3, -1) = \alpha_1(2, 3, -1) + \alpha_2(0, -1, 2) + \alpha_3(1, -3, 1)$$

$$= (2\alpha_1, 3\alpha_1, -\alpha_1) + (0, -\alpha_2, 2\alpha_2) + (\alpha_3, -3\alpha_3, \alpha_3)$$

$$= (2\alpha_1 + \alpha_3, 3\alpha_1 - \alpha_2 - 3\alpha_3, -\alpha_1 + 2\alpha_2 + \alpha_3)$$

$$\Rightarrow 2\alpha_1 + \alpha_3 = 1, \quad \ldots\ldots \text{(i)}$$

$$3\alpha_1 - \alpha_2 - 3\alpha_3 = 3, \quad \ldots\ldots \text{(ii)}$$

$$\text{and} \quad -\alpha_1 + 2\alpha_2 + \alpha_3 = -1 \quad \ldots\ldots \text{(iii)}$$

Multiplying (iii) by 3 and adding to (ii), we get that $\alpha_2 = 0$

Then (ii) and (iii) become, $3\alpha_1 - 3\alpha_3 = 3 \Rightarrow \alpha_1 - \alpha_3 = 1$

and $-\alpha_1 + \alpha_3 = -1 \Rightarrow \alpha_1 - \alpha_3 = 1$

Using (i) we get $3\alpha_1 = 2 \Rightarrow \alpha_1 = \frac{2}{3}$. Putting $\alpha_1 = \frac{2}{3}$ in (i)

We get, $\alpha_3 = \frac{-1}{3}$

Hence (1, 3, –1) is a linear combination of (2, 3, –1), (0,–1, 2) and (1, –3, 1)

where, $(1, 3, -1) = \frac{2}{3}(2, 3, -1) + 0(0, -1, 2) - \frac{1}{3}(1, -3, 1)$

### 3.3.2 THEOREM

Let $v_1, v_2, v_3, \ldots\ldots, v_n$ be a set of n vectors of a vector space V(F). The set $L[v_1, v_2, v_3, \ldots\ldots, v_n]$ of all vectors which are linear combinations of these vectors over F, forms a vector over the field F. The vector addition and scalar multiplication are defined as;

$$(\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + v_n v_n) + (\beta_1 v_1 + \beta_2 v_2 + \ldots\ldots + \beta_n v_n)$$

$$= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \ldots\ldots + (\alpha_n + \beta_n) v_n, \alpha_i, \beta_i \in F.$$

$$\alpha \cdot (\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n) = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \ldots\ldots + \alpha\alpha_n v_n.$$

$0_v = 0_F v_1 + 0_F v_2 + \cdots + 0_F v_n$. be the additive identity of $L[v_1, v_2, \ldots, v_n]$.

and $(-\alpha_1) v_1 + (-\alpha_2) v_2 + \ldots\ldots + (-\alpha_n) v_n$ be the additive inverse of $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n$.

### REMARKS

1. This vector space is called the space of linear manifolds of the set $\{v_1, \cdots, v_n\}$.
2. This is a space generated by the set $\{v_1, \cdots, v_n\}$ and written as $\langle v_1, v_2, \cdots, v_n \rangle$.
3. It is the smallest vector space containing this set $\{v_1, \cdots, v_n\}$ of vectors of V(F)
4. The student is advised to verify the properties of vector space.

Let $V = \langle v_1, v_2, \ldots\ldots, v_K \rangle = L[v_1, v_2, \ldots\ldots, v_n]$

There may be more than one subsets of a vector space V(F), which span V(F). e.g.

If $\{v_1 = (2, 0, 0),\ v_2 = (1, 3, 0),\ v_3 = (0, -2, 0)\} \subseteq R^3$.

We see that $v_1 = 2v_2 + 3v_3$

It implies that the space spanned by $\{v_2, v_3\}$ is the same as spanned by $\{v_1, v_2, v_3\}$

5. Consider the set $\{(1, 0), (0, 1)\}$ of vectors of $R^2$,

The space generated by this set is $R^2$.

i.e. $R^2 = \langle(1, 0), (0, 1)\rangle$,

because $\alpha, \beta \in R,\ \alpha(1, 0) + \beta(0, 1) = (\alpha, \beta) \in R^2$

Similarly the vector space $R^3 = \langle(1, 0, 0), (0, 1, 0), (0, 0, 1)\rangle$

For $\alpha, \beta, \gamma \in R$,

$$\alpha(1, 0, 0) + \beta(0, 1, 0) + \gamma(0, 0, 1) = (\alpha, \beta, \gamma) \in R^3$$

With the variation of $\alpha, \beta, \gamma \in R$, the whole space $R^3$ is generated.

### 3.3.3 LINEAR DEPENDENCE (L.D) OF VECTORS

Let V(F) be a vector space. The set $\{v_1, v_2, \ldots\ldots, v_m\}$ of non-zero vectors of V(F) is said to be L.D if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_m v_m = 0_v \qquad \ldots\ldots \text{(a)}$$

$$\Rightarrow \text{ not all } \alpha_i = 0_F,\ 1 \le i \le m$$

(It means that there is atleast one $\alpha_i \neq 0_F$).

Or the solution $(\alpha_1, \alpha_2, \cdots \alpha_m)$ of (a) is non-zero.

**EXAMPLE**

Is the set $\{(3, 0, -3), (-1, 1, 2), (4, 2, -2), (2, 1, 2)\}$ of $R^3$ L.D. over R?

Take $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$, such that

$$\alpha_1(3, 0, -3) + \alpha_2(-1, 1, 2) + \alpha_3(4, 2, -2) + \alpha_4(2, 1, 2) = 0_V$$

$$\Rightarrow \quad (3\alpha_1, 0, -3\alpha_1) + (-\alpha_2, \alpha_2, 2\alpha_2) + (4\alpha_3, 2\alpha_3, -2\alpha_3) + (2\alpha_4, \alpha_4, 2\alpha_4) = 0_V$$

$$\Rightarrow \quad (3\alpha_1 - \alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, -3\alpha_1 + 2\alpha_2 - 2\alpha_3 + 2\alpha_4) = (0,0,0)$$

By the equality of vectors, it implies then that ;

$$3\alpha_1 - \alpha_2 + 4\alpha_3 + 2\alpha_4 = 0 \quad \ldots\ldots \text{(i)}$$

$$\alpha_2 + 2\alpha_3 + \alpha_4 = 0 \quad \ldots\ldots \text{(ii)}$$

$$-3\alpha_1 + 2\alpha_2 - 2\alpha_3 + 2\alpha_4 = 0 \quad \ldots\ldots \text{(iii)}$$

Adding (i) with (iii) we get $\alpha_2 + 2\alpha_3 + 4\alpha_4 = 0$ ..... (iv)

Subtracting (ii) from (iv), we get, $3\alpha_4 = 0 \Rightarrow \alpha_4 = 0$

$\alpha_4 = 0$ gives (ii), (i) and (iii) by

$$\left.\begin{aligned} \alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + 4\alpha_3 &= 0 \\ -3\alpha_1 + 2\alpha_2 - 2\alpha_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots \text{(v)}$$

$(\alpha_2 + 2\alpha_3) + (-3\alpha_1 + 2\alpha_2 - 2\alpha_3) = -3\alpha_1 + 3\alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$

It thus gives (v) by

$$\left.\begin{aligned} \alpha_2 + 2\alpha_3 &= 0 \\ 2\alpha_2 + 4\alpha_3 &= 0 \\ -\alpha_2 - 2\alpha_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots \text{(vi)}$$

They all give that $\alpha_2 = -2\alpha_3 \Rightarrow 2\alpha_3 = -\alpha_1$

This holds for all such $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$.

$\alpha_1 = 2 = \alpha_2, \quad \alpha_3 = -1, \quad \alpha_4 = 0$

$\alpha_1 = 4 = \alpha_2, \quad \alpha_3 = -2, \quad \alpha_4 = 0$

$\alpha_1 = 6 = \alpha_2, \quad \alpha_3 = -3, \quad \alpha_4 = 0$ etc.

So the set is L.D.

### 3.3.4 LINEAR INDEPENDENCE (L.I) OF VECTORS

The subset $\{v_1, v_2, \ldots\ldots, v_m\}$ of a vector space V(F) over the field F is called Linearly Independent (L.I), if there exist $\alpha_i$. (i = 1, ——— ,m) $\in$ F such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_m v_m = 0_v \quad \ldots\ldots \quad (b)$$

$$\Rightarrow \quad \alpha_i = 0. \text{ for all i.}$$

or (b) has only the trivial solution $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots\ldots, 0)$.

### 3.3.5 UNIQUENESS OF THE EXPRESSION AS A LINEAR COMBINATION

Suppose that $\upsilon \in V(F)$ be a linear combination of $\upsilon_1, \upsilon_2, \ldots\ldots, \upsilon_n$.

Then, $\upsilon = \alpha_1\upsilon_1 + \alpha_2\upsilon_2 + \ldots\ldots + \alpha_n\upsilon_n$ , $\alpha_i \in F$.

We want to prove that $\upsilon$, as a linear combination of L.I vectors $\upsilon_1, \ldots\ldots, \upsilon_n$, can uniquely be expressed.

Suppose $\upsilon$ can be expressed in another way.

Then, $\upsilon = \beta_1\upsilon_1 + \beta_2\upsilon_2 + \cdots + \beta_n\upsilon_n$, $\beta_i \in F$, $\beta_i \neq \alpha_i$, for some or all i.

By the equality of vectors,

$$\alpha_1\upsilon_1 + \alpha_2\upsilon_2 + \ldots\ldots + \alpha_n\upsilon_n = \beta_1\upsilon_1 + \beta_2\upsilon_2 + \ldots\ldots + \beta_n\upsilon_n.$$

$$\Rightarrow \quad (\alpha_1 - \beta_1)\upsilon_1 + (\alpha_2 - \beta_2)\upsilon_2 + \ldots\ldots + (\alpha_n - \beta_n)\upsilon_n = 0_v.$$

For all i, $\upsilon_i \neq 0_v$ and L.I set $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$, it implies that $(\alpha_i - \beta_i)\upsilon_i = 0$, for each i.

$\Rightarrow \quad \alpha_i = \beta_i$, for each i, which provides contradiction. Hence the second expression is same as that of the first.

So the uniqueness of the linear combination of $\upsilon_1, \upsilon_2, \ldots\ldots, \upsilon_n$ follows.

**Noted that,**

Since $(0, 0) = 1(1, -1) + 1(-1, 1)$

$$= 2(1, -1) + 2(-1, 1) \ldots\ldots \text{ etc.,}$$

therefore (0, 0) is not uniquely expressed as a linear combination of the vectors (1, –1), (–1, 1), which form linearly dependent set.

### 3.3.6 REMARKS

1. The empty set $\phi$ of the vector space V(F) is always taken as Linearly Independent.

2. If a non-empty subset u of a vector space V(F) is not L.D then u is L.I in V(F) and the vice versa.

3. A subset $u = \{u_1, u_2, \cdots, u_n\}$ of a vector space V(F) is L.D if and only if there exists a vector $u_i \in u$ such that $u_i$ is a linear combination of other vectors of u.

4. A finite set of vectors of V(F) which includes zero vector is always L.D.

**PROOF**

Let $\{v_1, v_2, \cdots, v_m\}$ be a finite set of vectors of V(F) with $v_i = 0_v$, $1 \le i \le m$.

Then there exist scalars $0_1, 0_2 \cdots, \alpha_i, 0_{i+1}, \cdots, 0_m$ not all zero, such that

$$0_1 v_1 + 0_2 v_2 + \cdots\cdots + \alpha_i v_i (= 0_v) + \cdots\cdots 0_m \cdot v_m = 0$$

where $\alpha_i \neq 0$.

Hence the set $\{v_1, v_2, \cdots\cdots, v_i = 0_v, \cdots\cdots, v_m\}$ is L.D.

**COR.** Zero-vector is linearly dependent.

5. The singleton set $\{v\}$ of vector space V(F) is L.I iff $v \neq 0_v$.

**PROOF**

Suppose $\{v\}$ is L.I. Then $0_F \cdot v = 0_v$ and $v \neq 0_v$

Suppose $v \neq 0_v$

Then $\alpha \cdot v = 0_v$, only if $\alpha = 0_F$.

Therefore, $\alpha \cdot v = 0_v \Rightarrow \alpha = 0$

Hence the set $\{v\}$ is L.I.

### 3.3.7 THEOREM

For a vector space V(F) over the field F, let the set $V' = \{v_1, v_2, \cdots, v_t\}$ of vectors of V(F) be L.D over F. Then the extended set $V'' = \{v_1, v_2, \cdots, v_t, v\}$ is also L.D over F. ($v \neq 0_v \in V$).

**PROOF** The set V′ is L.D. Therefore, there exist $\alpha_1, \alpha_2, \cdots, \alpha_t \in F$ not all zero, such that,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_t v_t = 0_v$$

Let $\alpha_i$ be the non-zero coefficient of $v_i$ $1 \le i \le t$

Then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_i v_i + \cdots + \alpha_t v_t + 0_F v = 0_v$,

where $\alpha_i \neq 0_F$. Hence the set $V'' = \{v_1, v_2, \cdots, v_t, v\}$ is L.D.

### 3.3.8 THEOREM

In a vector space V(F), if the set $\{v_1, v_2, \cdots, v_m\}$ of (non-zero) m ($\geq 2$) vector is L.D. Then there exists an integer k, $2 \leq k \leq m$ such that $v_k$ is a linear combination of the vectors $\{v_1, v_2, \cdots, v_{k-1}\}$ and conversely.

**PROOF** Let the set $\{v_1, v_2, \cdots, v_m\}$ of V(F) be L.D. Suppose that there exists the first integer k from left for which the set $\{v_1, v_2, \cdots, v_k\}$ is L.D. Then there exist $\alpha_1, \alpha_2, \cdots, \alpha_k \in F$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_k v_k = 0_v$$

where $\alpha_k \neq 0$

Then $\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 + \cdots\cdots - \alpha_{k-1} v_{k-1}$

$$v_k = \frac{-\alpha_1}{\alpha_k} v_1 - \frac{\alpha_2}{\alpha_k} v_2 + \cdots\cdots \frac{-\alpha_{k-1}}{\alpha_k} v_{k-1}$$

$v_k$ is a linear combination of the vectors $\{v_1, v_2, \cdots\cdots, v_{k-1}\}$

Conversely, let $v_k = \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_{k-1} v_{k-1}$

$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_{k-1} v_{k-1} + 0 v_{k+1} + \cdots\cdots + 0 v_m$

Thus $\{v_1, v_2, \cdots, v_k, \cdots v_m\}$ is the set of L.D vectors.

**NOTE**

If $k = 1$, then we have $\alpha_1 v_1 = 0_v \cdot \alpha_1 \neq 0 \Rightarrow v_1 = 0_v$,

which provides contradiction to the fact that $\{v_1, \cdots, v_m\}$ is a set of non-zero vectors.

### 3.3.9 THEOREM

If $v_i$ $(i = 1, 2, \cdots, m) \in V(F)$ and $\alpha_i \in F (i = 1, 2, \cdots, m)$ such that the set $\{v_2 + r_2 v_1, v_3 + r_3 v_1, \cdots\cdots, v_m + r_m v_1\}$ is L.D, $(r_i \in F)$,

then the set $\{v_1, v_2, \cdots\cdots, v_m\}$ is L.D.

**PROOF** By definition there exist $\alpha_i$ $(i = 2, \cdots, m)$ in F, not all zero such that,

$$\alpha_2(v_2 + r_2 v_1) + \alpha_3(v_3 + r_3 v_1) + \cdots\cdots + \alpha_m(v_m + r_m v_1) = 0_v$$

$$\Rightarrow (\alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_m v_m) + (\alpha_2 r_2 + \alpha_3 r_3 + \cdots + \alpha_m r_m) v_1 = 0_v$$

Since, not all $\alpha_i$ are zero, therefore, the set $\{v_1, v_2, \cdots\cdots, v_m\}$ is L.D.

### 3.3.10 THEOREM

Any subset of a L.I set of vectors of vector space V(F) is L.I.

**PROOF** Suppose the set $\{v_1, v_2, \cdots\cdots v_p, v_{p+1}, \cdots v_r\}$ be L.I.

Then there exist $\alpha_1, \alpha_2, \cdots\cdots \alpha_r \in F$, all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p + \alpha_{p+1} v_{p+1} + \cdots + \alpha_r v_r = 0_v.$$

Consider a subset $\{v_1, v_2, \ldots\ldots, v_p\}$ of the given set $\{v_1, v_2, \cdots, v_p, v_{p+1}, \cdots, v_r\}$. Then there are scalars $\alpha_1, \alpha_2, \ldots\ldots \alpha_p$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots\ldots + \alpha_p v_p = 0_v$$

where all $\alpha_i$, $1 \leq i \leq p$, are zero.

Hence the set $\{v_1, v_2, \ldots\ldots, v_p\}$ is L.I.

### 3.3.11 EXAMPLES

1. Consider the vector space $R^3$ over R and the subset

   $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $R^3$ is L.I. because,

   $\alpha_1(1, 0, 0) + \alpha_2(0, 1, 0) + \alpha_3(0, 0, 1) = 0_v = (0, 0, 0)$

   $\Rightarrow \quad (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0, 0, 0)$

   $\Rightarrow \quad (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$

   $\Rightarrow \quad \alpha_1 = 0 = \alpha_2 = \alpha_3.$

2. Is the subset $\{(1, 1, 0), (2, 3, 0), (1, 0, 1)\}$ of $R^3$ L.I.?

   Suppose, there exist $\alpha_1, \alpha_2, \alpha_3 \in R$, such that

   $$\alpha_1(1, 1, 0) + \alpha_2(2, 3, 0) + \alpha_3(1, 0, 1) = (0, 0, 0)$$

   $$\Rightarrow \quad (\alpha_1, \alpha_1, 0) + (2\alpha_2, 3\alpha_2, 0) + (\alpha_3, 0, \alpha_3) = (0, 0, 0)$$

   $$\Rightarrow \quad (\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2, \alpha_3) = (0, 0, 0)$$

   $$\Rightarrow \quad \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 & \text{(i)} \\ \alpha_1 + 3\alpha_2 = 0 & \text{(ii)} \\ \alpha_3 = 0 & \text{(iii)} \end{cases}$$

   $\alpha_3 = 0$ from (iii). Put $\alpha_3 = 0$ in (i), solve it with (ii) simultaneously, and get $\alpha_1 = 0 = \alpha_2$.

   Hence the set is L.I.

3. Determine k, so that the set

   $\{(1, -1, k-1), (2, k, -4), (0, 2 + k, -8)\}$ of vectors of $R^3$ is L.D.

Since $R^3$ is a vector space over the field R, therefore, there exist

$\alpha_1, \alpha_2, \alpha_3 \in R$, such that

$$\alpha_1(1, -1, k-1) + \alpha_2(2, k, -4) + \alpha_3(0, 2 + k, -8) = (0, 0, 0)$$

$$\Rightarrow (\alpha_1, -\alpha_1, \alpha_1(k-1)) + (2\alpha_2, \alpha_2 k, -4\alpha_2) + (0, \alpha_3 (2+k), -8\alpha_3) = (0, 0, 0)$$

$$\Rightarrow \quad (\alpha_1 + 2\alpha_2, -\alpha_1 + k\alpha_2 + \alpha_3(2+k), \alpha_1(k-1) - 4\alpha_2 - 8\alpha_3) = (0, 0, 0)$$

By the equality of vectors,

$$\alpha_1 + 2\alpha_2 = 0 \quad \ldots\ldots \text{(i)}$$

$$-\alpha_1 + k\alpha_2 + \alpha_3(2 + k) = 0 \quad \ldots\ldots \text{(ii)}$$

$$\alpha_1(k-1) - 4\alpha_2 - 8\alpha_3 = 0 \quad \ldots\ldots \text{(iii)}$$

From (i), $\alpha_1 = -2\alpha_2$. (ii) and (iii) become,

$$+2\alpha_2 + k\alpha_2 + \alpha_3(2+k) = 0 = (k+2)\alpha_2 + \alpha_3(2+k) \quad \ldots\ldots \text{(iv)}$$

and $$-2k\alpha_2 + 2\alpha_2 - 4\alpha_2 - 8\alpha_3 = 0 = \alpha_2(-2-2k) - 8\alpha_3 \quad \ldots\ldots \text{(v)}$$

From (iv), $\alpha_3 = -\alpha_2$ and from (iv), (v) it becomes $\alpha_2(-2-2k) + 8\alpha_2 = 0$ which gives $k = 3$ $(\alpha_2 \neq 0)$

**Alternatively**, if the set $u = \{(1,-1,k-1), (2,k,-4), (0,2+k,-8)\}$ is L.D, then there exists a vector in u which is a linear combination of the other two vectors.

Say, there exist $\alpha, \beta \in F$, such that

$$(1,-1,k-1) = \alpha(2, k, -4) + \beta(0, 2+k, -8)$$

$$= (2\alpha, \alpha k, -4\alpha) + (0, \beta(2+k), -8\beta)$$

$$(1,-1,k-1) = (2\alpha, \alpha k + \beta(2+k), -4\alpha - 8\beta)$$

By the equality of vectors,

(i) $1 = 2\alpha \quad \Rightarrow \alpha = \frac{1}{2}$

(ii) $-1 = \alpha k + \beta(2 + k) \Rightarrow -1 = \frac{k}{2} + \beta(2 + k)$

$$\Rightarrow \frac{-(k+2)}{2} = \beta(2+k)$$

$$\Rightarrow \beta = \frac{-1}{2}$$

(iii) $k-1 = -4\alpha - 8\beta \Rightarrow k-1 = -4\left(\frac{1}{2}\right) - 8\left(\frac{-1}{2}\right)$

$$= -2 + 4 = 2$$

$$\Rightarrow k = 3$$

Hence the set u of $R^3$ is L.D for $k = 3$.

Note that, for every real value $k \neq 3$, the set

$\{(1, -1, k-1), (2, k, -4), (0, 2+k, -8)\}$ is L.I.

## EXERCISE 3.3

1. Simplify the following expressions;

   (a) $4(2, -1) + 3(1, 0) + 2(4, -3)$

   (b) $-3(2, 2) + 5(4, -3) + 7(1, 5) + 2(3, 3)$

   (c) $2(1, 1, 1) + 3(4, 4, 0) + (2, 1, -1) + (-1)(1, 1, 0)$

   (d) $2(1,3,5,-7) + (-2)(2,1,0,0) + (-3)(2,-2,1,-1) + (1,0,0,2)+3(2,-2,1,5)$

2. In $R^3$, show whether $L[(1, 2, 3), (0, 4, 1)] = L[(1, 6, 4), (1, -2, -2)]$?

3. In $R^3$, let $u = (1, 2, 3)$, $v = (0, 1, 2)$ and $w = (3, 1, -1)$. Then express $(8, 5, 2)$ as a linear combination of u, v and w if possible.

4. In the vector space $R^2$, show whether

   $$L[(2, 5), (1, 3)] = L[(1, 0), (0, 1)] \quad \text{or not?}$$

5. Show that the set $\{(1,3), (2,5)\}$ is L.I. but the set $\{(1,3),(2,6)\}$ is L.D. in $R^2$.

6. In each of the following cases, determine, whether the given subset S of $R^n$ is L.D or L.I,

   (a) $n = 3$, $S = \{(1, 1, 1), (1, -1, 3), (2, 5, -1)\}$

   (b) $n = 3$, $S = \{(1, 0, -1), (2, 3, 0), (0, 1, 4))\}$

   (c) $n = 4$, $S = \{(2, 1, 0, 1), (3, 1, 1, 4), (1, -1, 3, 1)\}$

   (d) $n = 4$, $S = \{(1, 3, -2, 4), (2, 4, -3, 6), (-3, 5, -1, 2)\}$

   (e) $n = 2$, $S = \{(-1, 3), (5, 9)\}$

(f) $n = 3,\ S = \{(6, 8, 1), (0, 0, 0), (-1, 1, -1)$

(g) $n = 4,\ S = \{(1, 0, 0, 0), (-1,3, 1,1), (1, 2, 3, 4), (7, 0, 7, 10)\}$

**7.** Prove that $\mathbb{C}^2$ is a vector space over $\mathbb{C}$ and $\mathbb{C}^2$ is also a vector space over R. Find a subset of $\mathbb{C}^2$ which is L.I over R but L.D over $\mathbb{C}$.

**8.** Show that the vectors (1, 0, 3, 1), (–1, 1, 0, 1), (2, 3, 0, 0) and (1, 1, 6, 3) over R are L.D. Can each of these vectors be expressed as a linear combination of the others ?

**9.** If two of the vectors $v_1, v_2, \ldots\ldots, v_n$ are equal, show that the vectors are L.D.

**10.** Let $(x_1, x_2)$ and $(y_1, y_2)$ be two vectors in $R^2$. Show that they are linearly independent if and only if $x_1y_2 - x_2y_1 \neq 0$ $\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0\right)$

**11.** For what value of $\lambda$, are the three vectors (1, 1, 2),(2,3,5) and (1,3,$\lambda$) of $R^3$ L.D.?

**12.** Let $\{v_1, v_2, \cdots, v_n\}$ be a set of vectors in $R^m$. Prove that they are L.I iff none of the vectors is a linear combination of the remaining vectors.

**13.** Show that the vectors (1, 0, 0), $(y_1, 1, 0)$ and $(z_1, z_2, 1)$ of $R^3$ are L.I. for any scalars $y_1, z_1, z_2$.

**14.** Show that the vectors $\left(3 + \sqrt{2}\ ,\ 1 + \sqrt{2}\right)$ and $\left(7,\ 1 + 2\sqrt{2}\right)$ in $R^2$ are L.D over R but L.I over Q, the rational field.

**15.** Show that the vectors (1– i, i) and (2, –1+i) in $\mathbb{C}^2$ are L.D over $\mathbb{C}$ but L.I over R.

**16.** Let V(F) be a vector space and S be a subset of V. Then prove that

(i) If S contains just one vector, then S is L.D iff the vector is the zero vector.

(ii) If S contains exactly two vectors, then S is L.D iff one of these vectors is a scalar multiple of the other.

(iii) If S contains more than two vectors then S is L.D iff some vector in S is a linear combination of one or more than one vectors in S.

## 3.4 SUBSPACE

### 3.4.1 DEFINITION

A subset U of a vector space V(F) forms a subspace of V(F), if

(i) $0_v \in U$

and (ii) U forms a vector space under the same operations of addition and scalar multiplication over the same field F as that of V.

Notation wise, $U \subseteq V(F)$ stands for "U as a subspace of V(F)".

A subspace U of $(V(F),+,\cdot)$ is a commutative additive subgroup of (V(F), +) and scalar multiplication is closed over U. {0}, V are subspaces of the vector space V(F), called the improper subspaces of V. All other subspaces are called proper. {0} subspace is also called null or trivial subspace. Keeping in view the hereditary properties of V(F) which naturally descend to every subset of V(F), the following theorem gives a short test for a subset to from a subspace of V(F).

### 3.4.2 THEOREM (Subspace Criterion)

Let U be a non–empty subset of the vector space V(F).

Then U is a subspace of V(F), if and only if,

(i) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$

and (ii) $\alpha \in F, u \in U \Rightarrow \alpha \cdot u \in U$

**PROOF** Suppose U satisfies (i) and (ii). Then the associative property $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$, $u_1, u_2, u_3 \in V(F)$ follows from the associative property of V(F). Similarly the commutative property of U(F) is clear.

From (ii) $\forall\ \alpha \in F,\ (-\alpha)u \in U \Rightarrow -(\alpha u) \in U$

Thus their sum $\alpha u - (\alpha u) = 0_v \in U$ (Identify property)

For every $u \in U$, $-u \in U$, because, $u + (-u) = 0_v$

The properties of scalar multiplication to U are hereditary. Hence U is a subspace of V(F).

Conversely, if U(F) is a subspace of V(F), then (i) and (ii) follow immediately by definition.

**COR.** A non–empty subset U(F) of a vector space V(F) is a subspace of V(F) iff;

$u_1, u_2 \in U$ , $\alpha$ , $\beta \in F \Rightarrow \alpha u_1 + \beta u_2 \in U$

For $\alpha = \beta = 1 \Rightarrow u_1 + u_2 \in U$

For $\beta = 0_F \Rightarrow \alpha u_1 \in U$

By definition, then U is a subspace of V(F). Conversely, U(F) is a subspace of V(F) $\Rightarrow$ $\alpha u_1 + \beta u_2 \in U$ (F).

### 3.4.3 EXAMPLES OF SUBSPACES

1. The set of all 3 x 3 matrices over the real field R forms a vector space under the operation of matrix addition and scalar multiplication of matrices.

i.e. $V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : a,b,c,d,e,f,g,h,i \in R \right\}$ is vector space over R.

(i) Subset $U = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} : a , b , c , d \in R \right\}$ forms a subspace of V(R).

(ii) The subset $U' = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a , b , c \in R \right\}$ of V(R) forms a subspace of the vector space of all 3 x 3 matrices over the field R.

(iii) The subset $U = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} : a, b, c, d, e, f \in R \right\}$ forms a subspace of V.

2. In the vector space $R^3 = \{(x,y,z) : x, y , z \in R\}$ over the field R of all real numbers

(i) The subset $\{(x,y,0) : x,y, \in R\}$ forms a subspace of $R^3$ over R.

(ii) The subset $\{(x,0,0) : x \in R\}$ of $R^3$ forms a subspace of $R^3$.

(iii) Every field is a vector space over itself.

3. The vector space of all the polynomials $p_n(x)$ in x of degree $\leq n$ over the field R, contains ($n \geq 1$):

(i) Subspace of all polynomial of $p_n(x)$ of degree $< n$.

(ii) There is a subspace $p_m(x)$ of $p_n(x)$, for every $m < n$. Polynomials of degree zero over the field R is just the field R itself.

### 3.4.4 THEOREM

Let U and V be two subspaces of a vector space W(F). Then $U \cap V$ is also a subspace of W(F).

**PROOF** If U and V are subspaces of vector space W(F), then $U \cap V$ is a subset of W(F).

Suppose, $t_1, t_2 \in U \cap V$. Then $t_1, t_2 \in U$ as well as V.

For $\alpha, \beta \in F$, $\alpha t_1 + \beta t_2 \in U$, because U is a subspace of W and $\alpha t_1 + \beta t_2 \in V$, because, V is a subspace of W, which implies that $\alpha t_1 + \beta t_2 \in U \cap V$, for all $\alpha, \beta \in F$.

Hence $U \cap V$ is a subspace of W(F).

**COR.** As $U \cap V$ is a subset of U as well as of V, which are vector spaces over the field F, $U \cap V$ is a subspace of U and as well as of V.

### 3.4.5 SUM OF SUBSPACES

Let S and T be two subspaces of a vector space V(F). Then

$$S + T = \{s + t : s \in S, t \in T\}$$

is a subset of V(F). S + T is called sum of the subspaces S and T of V(F).

For $S = 0_S$, $S + T = 0_S + T = T$

and for $T = 0_T$, $S + T = S + 0_T = S$

Thus S + T contains subspaces S and T and hence their union.

Infact, S + T is a space generated by $S \cup T$ which is the smallest such subspace of V(F). i.e. $S + T = \langle S \cup T \rangle$

### 3.4.6 THEOREM

Let S and T be two subspaces of a vector space V(F). Then S + T is a subspace of V(F).

**PROOF** Let $s_1 + t_1, s_2 + t_2 \in S + T$, $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Then

$$\begin{aligned}
(s_1 + t_1) + (s_2 + t_2) &= s_1 + (t_1 + s_2) + t_2 && \text{(associative property)} \\
&= s_1 + (s_2 + t_1) + t_2 && \text{(commutative property)} \\
&= (s_1 + s_2) + (t_1 + t_2) && \text{(associative property)}
\end{aligned}$$

As S and T are subspaces of V(F), therefore, $s_1 + s_2 \in S$ and $t_1 + t_2 \in T$.

Consequently, $(s_1 + t_1) + (s_2 + t_2) \in S + T$

As $0_S + 0_T = 0_V$, therefore $0_V \in S + T$

For $\alpha \in F$ $\quad \alpha(s_1 + t_1) = \alpha s_1 + \alpha t_1 \in S + T$

where $\alpha s_i \in S$ and $\alpha t_1 \in T$, as S and T are subspaces of V(F).

It thus proves that S + T is a subspace of V(F).

**COR.** For a finite family $W_1, W_2, \cdots, W_n$ of subspaces of a vector space V(F), $W_1 + W_2 + \cdots W_n = \sum_{i=1}^{n} W_i$ is a subspace of V(F), which is a sum of $W_1, W_2, \ldots, W_n$

or $$\sum_{i=1}^{n} W_i = \left\{ \sum_{i=1}^{n} x_i : x_i . \in W_i \right\}$$

### 3.4.7 DEFINITION

The space W = S + T is called direct sum of S and T, if each element $w \in W$ has a unique expression as a finite sum of the elements of S and T and is denoted by

$$W = S \oplus T$$

For a family of subspaces $W_1, W_1, \ldots, W_n$ of a vector space W(F).

if $W = W_1 \oplus W_1 \oplus \ldots \oplus W_n = \oplus \sum_{i=1}^{n} W_i$

then W is called the internal direct sum of $W_i$.

### 3.4.8 QUOTIENT SPACE

Let W be a subspace of V(F), then for a fixed element $v_1 \in V$, the subset

$[v_1] = \{v_1 + w : w \in W\}$ of V, is denoted by $v_1 + W$ and is called $v_1$ – coset of W in V(F).

If we carry on the process of forming cosets of W in V(F) until the whole space V(F) is obtained as union of the cosets of W in V(F). Then $V = \cup_i \{v_i + W\}$.

The set of all cosets by W in V is called the quotient set and is denoted by

$$V/W = \{v_i + W : v_i \in V\} = \{[v_i] : v_i \in V\}$$

The binary operations of addition and scalar multiplication on V/W be defined by

$$[u] + [v] = [u + v],$$

That is, $[u] + [v] = (u + W) + (v + W) = (u + v) + (W + W)$

$$= (u + v) + W$$

$$= [u + v]$$

$\alpha[u] = \alpha (u + W) = \alpha u + \alpha W = \alpha u + W = [\alpha u]$

For all $u, v \in V(F)$ and $\alpha \in F$.

### 3.4.9 DEFINITION

If $u, v \in V(F)$ and W is a subspace of V(F), then u is said to be congruent to v modulo W, denoted by $u \equiv v$ (modulo W), if $u - v \in W$.

### 3.4.10 THEOREM

Let W be a subspace of a vector space V(F). Then the relation of congruence (modulo W) is an equivalence relation on V(F).

**PROOF** The relation of congruence (modulo W) is reflexive, because

$$0 = u - u \in W, \text{ for every } u \in V(F).$$

For symmetry, suppose $u \equiv v$ (modulo W) which implies that $u - v \in W$.

$$\Rightarrow -(v - u) \in W$$

$$\Rightarrow v - u \in W, \text{ as } W \subset V(F)$$

which proves that $v \equiv u$ (modulo W).

If $u \equiv v$ (modulo W) and $v \equiv w$ (modulo W), then $u - v \in W$ and $v - w \in W$.

By the vector addition in W,

$$(u - v) + (v - w) = u - w \in W.$$

$$\Rightarrow u \equiv w \text{ (modulo W)},$$

which proves the transitivity of the congruence relation and hence the assertion of the theorem follows.

**COR. 1.** The cosets by W in V(F) form equivalence classes of V(F) and hence partition V(F).

$$\text{i.e.} \quad V = \cup\{v_i + W\} \text{ and } \quad (v_i + W) \cap (v_j + W) = \phi \ (i \neq j)$$

$$\text{or} \quad V = \cup_i [v_i], \qquad \phi = [v_i] \cap [v_j] \ (i \neq j)$$

**COR. 2.** If any element is common to any two cosets by W in V then those two cosets coincide.

### 3.4.11 THEOREM

Let V(F) be a vector space and $W, W_1, W_2$ be subspaces of V(F). Then

(a) $v \in u + W$, if and only if $u + W = v + W$.

(b) $v \in W$, if and only if $v + W = W$

(c) $0 \in W$, if and only if $u + W = W$

(d) If $u + W_1 \subseteq v + W_2$, then $W_1 \subseteq W_2$

(e) If $u + W_1 \subset v + W_2$, then $W_1 \subset W_2$

(f) If $u + W_1 = v + W_2$, then $W_1 = W_2$

**PROOF** The proof is left as an exercise.

### 3.4.12 THEOREM

If W is a subspace of a vector space V(F). Then the quotient set V/W forms a vector space over F under the operations of addition and scalar multiplication of cosets by W in V.

**PROOF** As $[u] + [v] = [u + v] \in \frac{V}{W}$,

$$\alpha\,[u] = [\alpha u] \in \frac{V}{W}$$

and $[0_v] = 0_v + W = W$ itself.

W is the "zero element" or additive identity of vector space $\frac{V}{W}$.

Then

(i) $[u] + ([v] + [w]) = [u] + ([u] + [w])$ (associative property)

(ii) $[u] + [v] = [u + v] = [v + u]$

$= [v] + [u]$ (Commutative property)

(iii) $[-u] = -[u]$ , for all $u \in V$, (inverse property)

which proves that $\left(\frac{V}{W}, +\right)$ is abelian group and the scalar multiplication observes the following properties ;

$\alpha([u] + [v]) = \alpha([u + v])$ (Addition of cosets)

$= [\alpha(u + v)]$ (Scalar multiplication)

$= [\alpha u + \alpha v]$ (Scalar multiplication)

$= [\alpha u] + [\alpha v]$ (Addition of cosets)

$= \alpha[u] + \alpha[v]$ (Scalar multiplication)

$(\alpha + \beta)[u] = [(\alpha + \beta) u] = [\alpha u + \beta u]$

$= [\alpha u] + [\beta u]$

$= \alpha [u] + \beta [u]$

$(\alpha \cdot \beta) [u] = [\alpha \beta u]$

$= [\alpha (\beta u)] = \alpha [\beta u] = \alpha (\beta [u])$

$1_F \cdot [u] = [1_F \cdot u] = [u]$ , $\forall$ u,v $\in$ V.

Thus V/W is a vector space on the same field F as that of V. $\frac{V}{W}$ is called the quotient space of V by W.

## EXERCISE 3.4

**1.** Determine whether the given collection of vectors is a subspace of $R^n$ for an appropriate n. If the sub collection fails to be a subspace, list the conditions that fail.

(a) $S = \{(x_1 , x_2 , x_3) \in R^3 : x_1 + x_2 - x_3 = 0 \}$

(b) $S = \{(x_1 ; x_2) \in R^2 : 2x_1 - 3x_2 = 1\}$

(c) $S = \{(x_1 , x_2 , x_3) \in R^3 : x_1 = 5x_3 , x_1 + x_2 - x_3 = 0 \}$

(d) $S = \{(x_1 , x_2 , x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$

2. Let A be an n x n matrix. Then show that
   (a) $S = \{u \in R^n : Au = u\}$ is a subspace of $R^n$.
   (b) $S = \{u \in R^n : Au = -2u\}$ is a subspace of $R^n$.

3. Show that the subset V of $R^2$ consisting of all possible linear combinations of (1, 1) and (1, 2) is a subspace.

4. Find all the subspaces of R.

5. Let R[x] be the vector space of all polynomials in x over R. Which of the following subsets of R [x] are subspaces of R[x]?
   (a) The set of all constant polynomials.
   (b) The set of all polynomials of degree 2, together with 0 polynomial.
   (c) The set of all polynomials which contain only even powers of x.

6. If V(F) is a vector space, prove that $\{0_v\}$ a subspace of V(F)

7. Which of the following are subspaces of $R^2$?
   (a) $\{(a, a) : a \in R\}$
   (b) $\{(a, a^2) : a \in R\}$
   (c) $\{(a, a + 1) : a \in R\}$
   (d) $\{(a^2, a) : a \in R\}$

8. Let V be the vector space of all real valued functions defined on R. Show whether the following subsets of V form subspaces of V;
   (a) The set of all even functions.
   (b) The set of all odd functions.

## 3.5 BASIS OF A VECTOR SPACE

### 3.5.1 THEOREM

Let V(F) be a vector space over F and the set $U = \{v_1, v_2, \ldots, v_n\}$ which spans the vector space V(F).

i.e. $V(F) = \langle v_1, v_2, \cdots, v_n \rangle$.

If set U contains r linearly independent vectors then $n \geq r$.

**PROOF** $V(F) = \langle v_1, v_2, \cdots, v_n \rangle$

Take a set $\{t_1, t_2, \cdots, t_r\}$ of L.I. vectors of U, therefore $t_1$ is a linear combination of $v_1, v_2, \cdots, v_n$.

So $V(F) = < t_1 , v_1, v_2, \dots , v_n >$

and the set $\{t_1, v_1, v_2, \dots , v_n\} = \sigma$ is L.D.

By theorem 3.3.7, some vector of $\sigma$ is a L. Combination of its preceeding one's.

Let $v_i$ be that vector such that $v_i \neq t_1$ because $\{t_1\}$ is L.I.

Remove $v_i$ from $\sigma$ and the set

$S = \{t_1, v_1, v_2, \dots v_{i-1}, v_{i+1}, \dots , v_n\}$ spans V(F).

Repeat this argument of pulling the vectors $v_i$ from the set spanning V(F) and inserting the vector $t_i$ (i = 1, 2, ..., r) r times and still the set

$U = \{t_1, t_2, \dots, t_r, v_{r+1}, \dots, v_n\}$ spans V(F).

So the spanning set U contains n vectors such than $n \geq r$, when r are L.I in U.

**COR.** Let V(F) be a vector space over the field F. The set of all linearly independent vectors of V(F) which spans the vector space V(F) contains minimum number of vectors which generate the space V(F).

### 3.5.2 DEFINITION

Let U be a non–empty subset of vector space V(F). Then U is said to be a basis of V(F) if

(i) U is L.I. subset of V(F)

and (ii) U spans the vector space V(F).

**EXAMPLES**

1. The set $\{e_1 = (1,0,0) , e_2 = (0,1,0) , e_3 = (0 , 0 , 1)$ forms a basis of the vector space $R^3$, called the standard basis of $R^3$.

2. The set $\{e_1 = (1,0) , e_2 = (0,1)\}$ forms a basis of the vector space $R^2$.Similarly $\{(1,0) , (1,1)\}$ is another basis of $R^2$, $\{e_1, e_2\}$ forms the standard basis of $R^2$.

3. The vector space of $m \times n$ matrices over R has m . n matrices in its basis each containing one non-zero entry at each of m . n places.

**REMARK** We know that more than one L.I. subsets of a vector space span a vector space, therefore, there can be more than one basis of a vector space.

## 3.5.3 DIMENSION OF A VECTOR SPACE

## 3.5.4 DEFINITION

The number m of vectors in the basis of a vector space V(F) is called the dimension of the vector space V(F).

If there are m finite number of vectors in the basis of the vector space V(F) then V(F) is called finite m–dimensional space otherwise V(F) is an infinite dimensional space. For brevity we write dim V = m, if m is the dimension of V(F).

## 3.5.5 REMARKS

If V(F) is an n–dimensional vector space, then

(a) Any (n + 1) vectors of V(F) are L.D.

(b) No set of (n – 1) vectors can span V(F).

(c) Any set of n L.I vectors of V(F) forms a basis of V(F).

(d) The null–subspace {0} has zero dimension.

(e) The dimension of $R^2$ is 2, $R^3$ is 3, and that of $R^n$ is n.

(f) The dimension of the vector space of all m × m matrices over R is $m^2$.

## 3.5.6 THEOREM

Let V(F) be an n–dimensional vector space with a basis $\{v_1, v_2, \ldots, v_n\}$. Then every vector $v \in V(F)$ can uniquely be expressed as a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

**PROOF** Since the vectors $v_1, v_2, \ldots, v_n$ form a basis of V(F) therefore, they generate V(F).

$$\text{i.e.} \quad < v_1, v_2, \ldots, v_n > = V(F)$$

Then every vector $v \in V(F)$ can be expressed as a linear combination of these vectors, that is, there exist $\alpha_i$ (i = 1, 2, 3, ⋯, n) ∈ F such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \quad \ldots\ldots \text{(i)}$$

If this expression is not unique then we may have $\beta_i \neq \alpha_i$ (i = 1 , 2, …, n) such that $v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ ..... (ii)

(i) and (ii) $\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

$$\Rightarrow (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \cdots + (\alpha_n - \beta_n) v_n = 0_v$$

As $v_i$(i = 1, 2, ⋯, n) are L.I over F, therefore,

$\alpha_i - \beta_i = 0$, for each i

$\Rightarrow$ $\alpha_i = \beta_i$ for each i, which provides contradiction.

$\Rightarrow$ The expression (ii) is the same as (i)

$\Rightarrow$ Uniqueness of the expression of v as a L. combination of $v_i$.

### 3.5.7 THEOREM

Let V(F) be a vector space of dimension $n > 1$. If $\{v_1, v_2, \cdots, v_r\}$ $(1 \leq r < n)$ is a set of L.I. vectors of V(F). Then there exist vectors $\{v_{r+1}, v_{r+2}, \cdots, v_n\}$ of V(F) such that $\{v_1, v_2, \cdots v_r, v_{r+1}, \cdots, v_n\}$ forms a basis of V(F).

Or

Every L.I. set of vectors of a vector space V(F) of dim $n > 1$ can be extended to be a basis of V(F).

Or

Every L.I. set of vectors of a vector space V(F) of dim $n > 1$ is a part of some basis of V(F).

**PROOF** Since $r < n$, therefore the set $\{v_1, v_2, \cdots, v_r\}$ is not a basis of V(F). So the set $\{v_1, v_2, \cdots, v_r\}$ can not generate the whole vector space V(F). Then there exists a vector $v_{r+1} \in V(F)$ such that $v_{r+1} \notin \langle v_1, v_2, \cdots, v_r \rangle$.

Now the set $\{v_1, v_2, \cdots, v_r, v_{r+1}\}$ is L.I.

If $r + 1 < n$, we can repeat this argument and continue untill we obtain the set $\{v_1, v_2, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ of n vectors which will be L.I. and will form a basis of V(F).

### 3.5.8 THEOREM

Any finite dimensional vector space V(F) contains a basis.

**PROOF** Let $\{v_1, v_2, \cdots, v_r\}$ be a subset of V(F) such that

$$\langle v_1, v_2, \cdots, v_r \rangle = V(F)$$

If the set $\{v_1, v_2, \cdots, v_r\}$ is L.I, then it forms a basis of V(F) (by definition).

If the set $\{v_1, v_2, \cdots, v_r\}$ is L.D, then there exists atleast one $v_i$ say, which is expressed as a linear combination of its preceeding vectors. Eliminating such a vector from the set $\{v_1, v_2, \cdots, v_r\}$ and labelling the vectors again we obtain the set $\{v_1, v_2, \cdots, v_{r-1}\}$ of $(r - 1)$ vectors.

i.e. $\langle v_1, v_2, \cdots, v_r \rangle = \langle v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_r \rangle$. Continuing this process we eventually arrive at the set $\{v_1, v_2, \cdots, v_n\}$, $1 \le n \le r$, which spans V(F) and is L.I. Hence it forms a basis of V(F).

**REMARK** The dimension of a vector space can not exceed the number of elements in a spanning set of this space.

### 3.5.9 THEOREM

All bases of a vector space V(F) contain the same number of vectors.

**PROOF** Suppose the sets A and B are different bases of a vector space V(F) of dim m and n respectively.

i.e. $A = \{v_1, v_2, \cdots, v_m\}$

and $B = \{u_1, u_2, \cdots, u_n\}$.

Since A spans V(F) and B is a linearly independent set of V(F), therefore,

$$m \ge n \quad \text{(by theorem 3.5.8)} \qquad \cdots\cdots \text{(i)}$$

Similarly B spans V(F) and A is L.I subset of V(F) which implies that

$$n \ge m \qquad \cdots\cdots \text{(ii)}$$

(i) and (ii) imply that $m = n$.

which proves the theorem.

**COR.** Let U be a proper subspace of a vector space V(F), then $\dim(U) < \dim(V)$.

### 3.5.10 THEOREM

Let S and T be two subspaces of a vector space V(F). If S and T be finite dimensional subspaces of V(F) then

$$\dim(S \cap T) + \dim(S + T) = \dim(S) + \dim(T).$$

**PROOF** As S and T are finite dimensional subspaces of V(F) of dimensions r+s and r+t respectively, $S \cap T$ is also finite dimensional and has a finite basis $\{u_1, u_2, \cdots, u_r\}$, if $S \cap T \neq \{0\}$. By theorem 3.5.7, the basis of $S \cap T$ can be extended to be a basis of S.

Let $B_1 = \{u_1, u_2, \cdots u_r, v_1, v_2, \cdots v_s\}$ be a basis of S.

Similarly, $B_2 = \{u_1, u_2, \cdots, u_r, w_1, w_2, \cdots, w_t\}$ be an extended basis of T, the super space of $S \cap T$.

The set $\{u_1, u_2, \cdots u_r, v_1, v_2, \cdots v_s, w_1, w_2, \cdots w_t\}$

Contains $(r + s + t)$ distinct vectors and spans the space $S + T$.

Thus dim $(S + T) \leq (r + s + t)$

Suppose that there exist scalars $a_1, a_2, \cdots, a_r, b_1, b_2, \cdots b_s$ and $c_1, c_2, \cdots c_t \in F$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_ru_r + b_1v_1 + b_2v_2 + \cdots + b_sv_s + c_1w_1 + c_2w_2 + \cdots + c_tw_t = 0$$

Then,

$$a_1u_1 + a_2u_2 + \cdots + a_vu_v + b_1v_1 + b_2v_2 + \cdots + b_sv_s = -(c_1w_1 + c_2w_2 + \cdots + c_tw_t)$$

$$\text{If } x = a_1u_1 + a_2u_2 + \cdots + a_ru_r + b_1v_1 + b_2v_2 + \cdots + b_sv_s = -(c_1w_1 + c_2w_2 + \ldots + c_tw_t) \quad \ldots(i)$$

then $x \in S \cap T$. x is then linear combination of the basis of $S \cap T$.

Thus there exist $d_1, d_2, \cdots, d_r \in F$ such that,

$$x = d_1u_1 + d_2u_2 + \cdots + d_ru_r \qquad \ldots\ldots \text{(ii)}$$

Subtracting (i) from (ii), we get

$$x - x = 0 = (d_1 - a_1)\,u_1 + (d_2 - a_2)\,u_2 + \cdots (d_r - a_r)\,u_r - b_1v_1 - b_2v_2 - \cdots - b_sv_s$$

$$= d_1u_1 + d_2u_2 + \cdots + d_ru_r + c_1w_1 + c_2w_2 + \cdots + c_tw_t \qquad \ldots\ldots \text{(iii)}$$

As $B_2$ is a basis of T, therefore, $d_i = 0 = c_j$, $1 \leq i \leq r$, $1 \leq j \leq t$.

Similarly, $B_1$ is a basis of S therefore, $a_i - d_i = 0 = b_k$, $1 \leq i \leq r$, $1 \leq k \leq s$

$a_i - d_i = 0 \Rightarrow a_i = d_i$, for each i. Hence $a_i = 0$.

Thus the set $\{u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s, w_1, w_2, \cdots, w_t\}$ is L.I and hence forms a basis of $S + T$. dim $(S + T) = r + s + t = (r + s + t) + r - r$

$$= (r + s) + (r + t) - r.$$

Hence dim $(S + T)$ = dim $(S)$ + dim $(T)$ – dim $(S \cap T)$

### 3.5.11 THEOREM

Let a vector space V(F) be the internal direct sum of its subspaces $W_1$ and $W_2$. Then,

(i) $W_1 \cap W_2 = \{0\}$

(ii) $\dim(W_1) + \dim(W_2) = \dim(V)$

**PROOF**

(i) If $V = W_1 \oplus W_2$ and $0 \neq u \in W_1 \cap W_2$

then $u \in W_1$ and $u \in W_2$ both. Thus there exists $v \in V(F)$ such that

$v = u + u'$, if $u \in W_1$ and $u' \in W_2$

and $v = u'' + u$, if $u \in W_2$ and $u'' \in W_1$

There two expressions are same as V is the direct sum of $W_1$ and $W_2$.

Thus $u + u' = u'' + u$.

$\Rightarrow u + u' = u + u''$ (Commutative property of V)

$\Rightarrow u' = u''$, which provides contradiction.

Hence $u = 0$ and $W_1 \cap W_2 = \{0\}$.

(ii) By the theorem 3.5.10,

$\dim V = \dim W_1 + \dim W_2$

## EXERCISE 3.5

**1.** Mark the following statements true or false. Give a brief explanation in support of your answer.

(i) If the set $\{u_1, u_2, \cdots, u_n\}$ is L.I. in a vector space then set $\{u_2, u_3, \cdots, u_n\}$ is also L.I.

(ii) Any L.I. set of vectors in $R^m$ is part of a basis of $R^m$.

(iii) Zero-vector is a linear combination of any collection of vectors in a vector space.

(iv) Every set of L.I. vectors that spans a subspace of a vector space forms a basis of that subspace.

(v) It is possible to find five vectors that span $R^6$.

(vi) If $\dim V = n$, then n is the maximum number of L.I. vectors in a vector space V(F).

(vii) If U and V are subspaces of $R^m$ and $\dim U = \dim V$, then $U = V$.

(viii) The dimension of a subspace of a vector space is the maximum number of L.I. vectors that can be contained in the subspace.

(ix) If U is a subspace of finite dimensional vector space W(F), then $\dim(W/U) = \dim W - \dim U$.

2. If V(F) is a vector space of dimension n, then prove that,
   (i) Every basis of V has n elements.
   (ii) Every L.I subset of V containing n element is a basis of V.
   (iii) Every subset of V containing n elements which generates V is a basis of V.
   (iv) No L.I subset of V has more than n elements.
   (v) No subset of V which generates V has less than n elements.
   (vi) Every L.I subset of V is a subset of a basis of V.
   (vii) Every subset of V which generates V contains a basis of V.

3. Let V(F) be a vector space of dimension n. Let S be a subset of V containing n vectors. Then S generates V if and only if S is L.I.

4. Let V(F) be a finite dimensional vector space and S be a L.I subset of V. Then there is a basis of V which contains S.

5. Prove that $R^m$ can not contain more than m L.Independent vectors.

6. Examine whether the following vectors form a basis of $R^3$.
   (i) $\{(1, 2, -1), (0, 3, 1)\}$
   (ii) $\{(2, 4, -3), (0, 1, 1), (0, 1, -1)\}$
   (iii) $\{(2, 0, 1), (1, 1, -1), (0, 0, 1)\}$.

7. Let U and V be two subspaces of W(F) such that $W = U + V$ and W is not internal sum of U and V. Show that $U \cap V$ contains some non-zero vectors.

8. (i) Find the basis of the subspace of $R^4$, generated by
   $v_1 = (1, 1, 2, 0)$, $v_2 = (2, 1, 0, 2)$
   $v_3 = (1, 2, 3, 4)$, $v_4 = (0, 4, 5, 2)$
   (ii) Find the real $a$ such that $(a, 1, 2, 0) \in L[v_1, v_2, v_3]$.

9. Find a basis of the subspace of $R^3$, generated by the vectors
   $v_1 = (1, 0, -1)$, $v_2 = (1, 2, 1)$ and $v_3 = (0, -3, 2)$

10. Find the basis of the vector space $V = F[x]$ of all polynomials in x over F.

**11.** Extend the set $\{(2, 1, 4, 3), (2, 1, 2, 0)\}$ to form a basis of $R^4$.

**12.** Examine whether the subset $\{(1, 1), (3, 1)\}$ of $R^2$, forms a basis of $R^2$.

**13.** Find the dimensions of the subspaces of $R^4$, spanned by;
(i) $\{(1, -2, 3, 1), (1, 1, -2, 3)\}$
(ii) $\{(-1, -1, 5, 0), (0, 0, 0, 1)\}$
(iii) $\{(1, 0, 2, 3), (2, 0, 4, 6)\}$

**14.** Let set B be a basis for a vector space V(F). Show that no proper subset of B is a basis for V(F).

**15.** Let A and B be two subsets of a vector space V(F) such that $A \subset B$. Prove that the subspace of V generated by A is properly contained in the subspace generated by B or $\langle A \rangle = \langle B \rangle$.

**16.** Let $W_1$, $W_2$ be two finite dimensional subspaces of a vector space W(F), such that $W_1 \subset W_2$. Then $W_1 = W_2$ if and only if dim $W_1$ = dim $W_2$.

**17.** Let $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\}$ be a vector space of dimension $2^2$. Then,

(i) Find the basis of $M_2$.

(ii) Prove that $U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in R \right\}$

and $V = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} : a \in R \right\}$ are subspaces of $M_2$. Find also their dimensions.

(iii) Show that $M_2$ is internal sum of U and V.

## 3.6 LINEAR TRANSFORMATIONS

In studying mathematical systems, we are often concerned with those mappings from a system into itself or into another similar system which preserve certain properties of the system. This section will be concerned with mappings of one vector space into another vector space which preserve the vector space operations of vector addition and scalar multiplication. Such a mapping is called a linear transformation. In the case where the vector spaces are finite dimensional, there exists a natural correspondence between the linear transformations and matrices. This correspondence will be established in the next chapter.

### 3.6.1 DEFINITION

Let U(F) and V(F) be two vector spaces over the same field F, and $T : U \to V$ be a mapping from U into V. Then T is called a linear transformation if it preserves the vector addition,

i.e $T(u_1 + u_2) = T(u_1) + T(u_2) = v_1 + v_2$ ..... (i)

for all $u_1, u_2 \in U$, $T(u_1) = v_1$ and $T(u_2) = v_2 \in V$

and scalar multiplication.

i.e. $\forall \alpha \in F$, $u \in U$, $T(u) \in V$,

$T(\alpha u) = \alpha T(u)$. ..... (ii)

Note that, T is an additive group-homomorphism, which preserves the scalar multiplication.

T is called non-singular or invertable if T is (1 – 1) and onto, otherwise T is singular.

### 3.6.2 EXAMPLES

1. The mapping $T : R^3 \longrightarrow R^2$ from $R^3$ into $R^2$, defined by

$$T(x, y, z) = (x - z, y - z)$$

is a linear transformation, because,

$$\begin{aligned} T[(x, y, z) + (x_1, y_1, z_1)] &= T(x + x_1, y + y_1, z + z_1) \\ &= (x + x_1 - z - z_1, y + y_1 - z - z_1) \\ &= (x - z + x_1 - z_1, y - z + y_1 - z_1) \\ &= (x - z, y - z) + (x_1 - z_1, y_1 - z_1) \\ &= T(x, y, z) + T(x_1, y_1, z_1) \end{aligned}$$

for all $(x, y, z), (x_1, y_1, z_1)$ in $R^3$.

and, for all $\alpha \in R$, $T\,\alpha(x, y, z) = T(\alpha x, \alpha y, \alpha z) = (\alpha x - \alpha z, \alpha y - \alpha z)$

$= \alpha \cdot (x - z, y - z) = \alpha T(x, y, z)$

2. Let $T : \mathbb{C} \to \mathbb{C}$ be a mapping over the complex space $\mathbb{C}$, defined by $T(z) = \bar{z}$, for all $z \in \mathbb{C}$. Then T is a linear transformation, because,

$$\begin{aligned} T(z + z_1) &= \overline{z + z_1} \\ &= \bar{z} + \bar{z}_1 \\ &= T(z) + T(z_1) \end{aligned}$$

and $T(\alpha z) = (\overline{\alpha z}) = \bar{\alpha} \cdot \bar{z} = \alpha T(z)$, for all $\alpha \in R$.

3. Let $T : R^2 \to R^3$ be a mapping from $R^2$ into $R^3$,

defined by $T(x, y) = (x + 1, 2y, x + y)$,

then T is not a linear transformation, because,

$$T[(x, y) + (x_1, y_1)] = T(x + x_1, y + y_1)$$

$$= (x + x_1 + 1, 2(y + y_1), x + x_1 + y + y_1)$$

$$\neq T(x,y) + T(x_1,y_1) = (x+1,2y,x+y)+ (x_1+1,2y, x_1+y_1)$$

$$= (x+x_1+2,2(y+y_1), x+x_1+y+y_1)$$

$$\text{Also } T\,\alpha(x, y) = T(\alpha x, \alpha y) = (\alpha x + 1, 2\alpha y, \alpha(x + y))$$

$$= \alpha\left(x + \frac{1}{\alpha}, 2y, x + y\right)$$

$$\neq \alpha \cdot T(x, y)$$

### 3.6.3 THEOREM

Let U and V be two vector spaces over the same field F and $T : U \to V$ be a linear transformation, then

(i) $T(0_u) = 0_v$

(ii) $T(-u) = -T(u)$, for all $u \in U$

**PROOF** Since T is an additive group-homomorphism therefore,

(i) and (ii) follow from theorem 2.4.7.

### 3.6.4 THEOREM

Let V and W be any two vector spaces over the field F and $T : V \to W$ be a linear transformation. Then

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$$

for all $v_1, v_2, \cdots, v_n \in V$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$.

**PROOF** $T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = T(\alpha_1 v_1) + T(\alpha_2 v_2 + \cdots + \alpha_n v_n)$

$$= \alpha_1 T(v_1) + T(\alpha_2 v_2) + T(\alpha_3 v_3 + \cdots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$$

Note that, T preserves linear combinations of the vectors. This is one of the reasons why, we call such mappings linear transformations. In other words, linear transformations preserve L.Dependence but do not, in general, preserve linearly independence.

### 3.6.5 THEOREM

Let $T : V(F) \to V'(F)$ be a linear transformation. Then,

(a) If S is any subspace of V then T(S) is a subspace of V′.

(b) If S′ is any subspace of V′ then the pre-image of S′ under T is a subspace of V.

**PROOF**

(a) Let $s_1, s_2 \in S$ and $T(s_1), T(s_2) \in T(S)$. If S is a subspace of V, then, for $\alpha, \beta \in F$, $\alpha s_1 + \beta s_2 \in S$,

and $T(\alpha s_1 + \alpha s_2) = \alpha T(s_1) + \beta T(s_2) \in T(S)$

Hence T(S) is a subspace of V′.

(b) If S′ is a subspace of V′(F), then, for every $\alpha, \beta \in F$,

$\alpha s'_1 + \beta s'_2 \in S'$, for $s'_1, s'_2 \in S'$.

Take $s_1 \in T^{-1}(S')$, $s_2 \in T^{-1}(S')$ such that $s_1 = T^{-1}(s'_1)$,

$s_2 = T^{-1}(s'_2)$, then $T(s_1) = s'_1$ and $T(s_2) = s'_2$.

For $\alpha, \beta \in F$,

$$T(\alpha s_1 + \beta s_2) = \alpha T(s_1) + \beta T(s_2)$$
$$= \alpha s'_1 + \beta s'_2$$
$$\text{or } \alpha s_1 + \beta s_2 \in T^{-1}(\alpha s'_1 + \beta s'_2)$$

As S′ is a subspace of V′, $\alpha s'_1 + \beta s'_2 \in S'$

and so $\alpha s_1 + \beta s_2 \in T^{-1}(S')$, for all $s_1, s_2 \in T^{-1}(S')$ and $\alpha, \beta \in F$.

Hence $T^{-1}(S')$ is a subspace of V(F).

### 3.6.6 DEFINITION

Let $T : V \to W$ be a linear transformation from a vector space V into another vector space W over the same field F. The subset of V consisting of all the vectors of V which are mapped by T onto the zero-vector of W, is called the kernel of the linear transformation T and denoted by Ker T.

i.e. $\text{Ker } T = \{v \in V : T(v) = 0_W\}$

### 3.6.7 THEOREM

Let $T : W(F) \to W'(F)$ be a linear transformation from vector space W(F) into another space W′(F). Ker T is a subspace of W.

**PROOF** Let $w_1, w_2 \in$ Ker T. Then $T(w_1) = 0_{W'} = T(w_2)$

Take $\alpha, \beta \in F$. Then

$$T(\alpha w_1 + \beta w_2) = \alpha T(w_1) + \beta T(w_2)$$

$$= \alpha \cdot 0_{W'} + \beta \cdot 0_{W'}$$

$$= 0_{W'} + 0_{W'} = 0_{W'}$$

which implies that $\alpha w_1 + \beta w_2 \in$ Ker T, for all $\alpha, \beta \in F$ and $w_1, w_2 \in W$.

Hence Ker T is a subspace of W.

The following theorem states the relationship of the Kernel and the image of a linear transformation T via its one-to-one and onto properties.

### 3.6.8 THEOREM

Let V and W be vector spaces and $T : V \to W$ be a linear transformation. Then

(i) T is (1 – 1) iff. Ker T = $\{0_V\}$, and

(ii) T is onto iff T(V) = W.

**PROOF** (i) Let us suppose that T is (1 – 1) and $0 \neq u \in$ Ker T.

Then $T(0) = T(u) \Rightarrow 0 = u$, which contradicts with the assumption that $u \neq 0$.

Hence Ker T = {0}

Conversely, suppose that Ker T = {0}, and $v_1, v_2 \in V$.

where $T(v_1) = T(v_2)$

$\Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 = 0$, by our supposition.

This implies that $v_1 = v_2$

Hence T is (1 – 1).

(ii) is left as an exercise.

### 3.6.9 THEOREM

Let V and W be two vector spaces over the field F with V finite n-dimensional and

$T : V \to W$ be a linear transformation. Then

$$\dim (\text{Ker } T) + \dim (T(V)) = \dim V$$

**PROOF** Let $\{v_1, v_2, \cdots, v_k\}$ be a basis for Ker T. The basis of subspace Ker T can be extended to a basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ of V. If we can show that $\{T(v_{k+1}), \cdots, T(v_n)\}$ is a basis of T(V) then we are done.

For each vector $v \in V$, there exist scalars $\alpha_1, \alpha_2, \cdots \alpha_n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n$$

Then, $T(v) = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \cdots + \alpha_n v_n)$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_k T(v_k) + \cdots\cdots + \alpha_n T(v_n)$$

$$= 0 + 0 + \cdots\cdots + 0 + \alpha_{k+1} T(v_{k+1}) + \cdots\cdots + \alpha_n T(v_n)$$

as, $v_1, v_2, \cdots\cdots, v_k \in$ Ker T.

$$= \alpha_{K+1} T(v_{k+1}) + \cdots\cdots + \alpha_n T(v_n)$$

Hence every vector $T(v) \in T(V)$ is a linear combination of $\{T(v_{k+1}), \cdots T(v_n)\}$

i.e. $\langle T(v_{k+1}), \cdots\cdots, T(v_n) \rangle = T(V)$.

To show that the set $\{T(v_{k+1}), \cdots\cdots, T(v_n)\}$ is L.I, suppose that, for $c_{k+1}$, $c_{k+2}, \cdots\cdots, c_n \in F$,

$$c_{k+1} T(v_{k+1}) + c_{k+2} T(v_{k+2}) + \cdots\cdots + c_n T(v_n) = 0_W$$

$$\Rightarrow \quad T(c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots\cdots + c_n v_n) = 0_W$$

$$\Rightarrow \quad c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots\cdots + c_n v_n \in \text{Ker } T.$$

Since $\{v_1, v_2, \cdots, v_k\}$ is a basis of Ker T, therefore,

$$c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

for some $c_1, c_2, \ldots\ldots, c_k \in F$.

Thus $(-c_1)\, v_1 + (-c_2)v_2 + \cdots + (-c_k)\, v_k + c_{k+1}v_{k+1} + \cdots + c_n v_n = 0_W$.

Since set $\{v_1, v_2, \ldots\ldots, v_n\}$ is L.I, therefore,

$$c_1 = c_2 = \ldots\ldots = c_k = c_n = 0_F$$

Inparticular, $c_{k+1} = \ldots\ldots = c_n = 0_F$

Thus the set $\{T(v_{k+1}), \ldots\ldots, T(v_n)\}$ is linearly independent,

which forms a Basis of T(V).

Hence $\dim(\text{Ker } T) + \dim(T(V)) = k + (n-k) = n$

$= \dim(V)$.

### 3.6.10 DEFINITION

A linear map $T : V \to W$ is invertable if there exists a linear map $T_1 : W \to V$ such that $T_1 \cdot T(v) = v$, for all $v \in V$.

and $T \cdot T_1(w) = w$, for all $w \in W$.

The linear map $T_1$, if it exists, is unique and is called the inverse of T. The inverse of T is denoted by $T^{-1} = T_1$.

### 3.6.11 DEFINITION

A linear transformation of a vector space V into V itself is said to be a linear operator on V.

## EXERCISE 3.6

1. Let $T : R^3 \to R^2$ be a vector function. Then determine, whether T is a linear transformation or not if defined by;

(i) $T(a_1, a_2, a_3) = (a_2, 0)$

(ii) $T(a_1, a_2, a_3) = (a_1, a_2)$

(iii) $T(a_1, a_2, a_3) = (0, a_1)$

(iv) $T(a, b, c) = (a + b, a + c)$

(v) $T(a, b, c) = (a + b, a - c)$

(vi) $T(a, b, c) = (|a|, 0)$

(vii) $T(a, b, c) = (a, 1)$

for all $(a_1, a_2, a_3), (a, b, c) \in R^3$.

2. Let $T : R^3 \to R^3$ be a mapping. Determine, whether T is a linear operator or not, if defined by;

(i) $T(x, y, z) = (x + y, 0, 0)$

(ii) $T(x, y, z) = (x + y, y, z)$

(iii) $T(x, y, z) = (x + 1, y, z)$

(iv) $T(x, y, z) = (x^2, y^2, z^2)$.

(v) $T(x, y, z) = (x-y, y-z, z-x)$

for all $(x, y, z) \in R^3$.

3. Let $T : R^2 \to R^2$ be a mapping. Determine, whether T is a linear operator or not, if defined by;

(i) $T(x, y) = (0, y)$

(ii) $T(x, y) = (x, y)$

(iii) $T(x, y) = (-y, x)$

(iv) $T(x, y) = (x + 1, y + 1)$

(v) $T(x, y) = (x - y, y)$

for all $(x, y) \in R^2$.

4. Show that T is a linear operator of V(F) whenever defined by

$T(v) = v$, for all $v \in V$.

5. Let $T : V(F) \to V'(F)$ be a linear transformation. Then, prove that,

(i) $T(0_V) = 0_{V'}$

(ii) $T(u - v) = T(u) - T(v)$, $u, v \in V$.

(iii) T(W) is a subspace of V', if W is a subspace of V.

6. Prove that if a linear transformation from V(F) into V'(F) is (1 – 1), then it preserves the linear independence of vectors.

7. Let $T : V(F) \to V'(F)$ be a linear transformation. If W is a subspace of V of dimension k then prove that dim $T(W) \leq k$.

8. If V and W have the same dimensions and $T : V(F) \to W(F)$ be a linear transformation. Prove that T is (1 – 1) if and only if T is onto W.

9. Let $T : \mathbb{C} \to \mathbb{C}$ be a mapping defined by,

$$T(x + iy) = iy$$

then show that T is a linear operator of the complex space $\mathbb{C}$ over the real field R. Find also its kernel.

10. Is the mapping defined in Q.9, linear operator of the complex space $\mathbb{C}$ over the complex field?

# 4

# MATRICES, DETERMINANTS AND THEIR APPLICATIONS

## 4.1 MATRICES

First, the readers will be made familiar, with the entity of a matrix which is constructed from the elements of a field.

### 4.1.1 DEFINITION OF A MATRIX

Let (F, +,·) be a field. The horizontal and vertical arrangement of the elements of F, which is enclosed by the square bracket, is called a matrix over the field F.

i.e.

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1j} & \vdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \vdots & \vdots & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \vdots & a_{ij} & \vdots & a_{in} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \vdots & a_{mj} & \vdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in F.$$

$$1 \le i \le m, \ 1 \le j \le n.$$

The matrix entries $a_{ij}$, $1 \le i \le m$, $1 \le j \le n$ are called elements of the matrix M.

A horizontal line made by the entries of F in the matrix forms a **row of the matrix**. The vertical line so formed will be named as a **column of the matrix.** So the matrix M has m rows and n columns. The entry $a_{ij}$ is placed in the ith row and jth column of the matrix M.

We shall read such a matrix as m x n or (m by n) matrix or a matrix M of order m · n.

If the number n of rows of a matrix is equal to the number of columns of this matrix then the matrix shall be called a **square matrix** of order $n^2$.

If the number m of rows is different from that of its n columns then the matrix is called **rectangular** of order m · n or mn.

NOTATION For brevity, if M is a m x n rectangular matrix then we write

$$M = [a_{ij}]_{(m, n)} \quad \text{or } [a_{ij}]_{mxn}$$

If a matrix M consists of a single row with n columns in the form as,

$$M = [a_{11}, a_{12}, \ldots\ldots, a_{1n}]$$

then M is called a **row matrix** or a 1 x n matrix.

If a matrix M consists of m rows and a single column in the form as,

$$M = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

then M is called a **column matrix** or a m x 1 matrix.

$[a_{11}]$ is a 1×1 matrix. Since, for every element $a_{11}$ in F, there is 1×1 matrix $[a_{11}]$.

therefore we shall denote a 1 x 1 matrix by $a_{11}$ instead of $[a_{11}]$.

For a square matrix $[a_{ij}] = M$ of order $n^2$, $1 \le i \le n$, $1 \le j \le n$, the entries $a_{ii}$ form the **leading diagonal** or **the principal diagonal** of the matrix M.

### 4.1.2 DEFINITION

Let $M = [a_{ij}]$ be an m x n matrix over a field F. Then the matrix $M^t = [a_{ji}]_{nxm}$ is called the transpose of M. The rows and columns are interchanged to produce $M^t$ from M.

For example, if

$$M = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 8 & 2 & 3 & 1 \\ 9 & 7 & 2 & -1 \end{bmatrix}$$ is a $3 \times 4$ matrix

then,

$$M^t = \begin{bmatrix} 0 & 8 & 9 \\ 2 & 2 & 7 \\ 3 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix}$$ is a $4 \times 3$ matrix

**REMARK** For a square matrix M, the leading diagonal remains unchanged by its transposition, that is, if $M = [a_{ij}]$ and $M^t = [a_{ji}]$, then $a_{ij} = a_{ji}$ for $i = j$

$$M = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 8 \\ 1 & 8 & 2 \end{bmatrix} \text{ and } M^t = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 8 \\ -1 & 8 & 2 \end{bmatrix}$$

### 4.1.3 DEFINITION

Let $M = [a_{ij}]$ be a matrix over the complex field $\mathbb{C}$. Then the matrix $\overline{M} = [\overline{a}_{ij}]$ is called the **conjugate** of M. The entries of M are replaced by their respective complex conjugates to produce $\overline{M}$.

For example, if

$$M = \begin{bmatrix} 1 & i & -i \\ 1+i & 1 & 2i \\ 2-i & 2 & 3i \end{bmatrix}, \text{ then}$$

$$\overline{M} = \begin{bmatrix} 1 & -i & i \\ 1-i & 1 & -2i \\ 2+i & 2 & -3i \end{bmatrix}$$

is the conjugate of M.

**Note.** If M is real, then $\overline{M} = M$.

### 4.1.4 DEFINITION (EQUALITY)

Two matrices $M = [a_{ij}]$ and $N = [b_{ij}]$ are said to be equal and denoted by $M = N$ iff;

(i) They have the same number of rows and same number of columns.

and (ii) Each element of M is equal to the corresponding element of N.

i.e. $a_{ij} = b_{ij}$, for each i and j

### 4.1.5 DEFINITION

A matrix $S = [a_{ij}]$ is said to be zero-matrix if every entry of the matrix S is zero.

i.e.

$$[0] = \begin{bmatrix} 0 & 0 & 0 :: 0 \\ 0 & 0 & 0 :: 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 :: 0 \end{bmatrix}$$

The order of the zero-matrix is undefined and its order can be taken as one desires.

### 4.1.6 DEFINITION

Let $M = [a_{ij}]$ be an m x n matrix. Then the **negative** of the matrix M is defined to be as,

$$-M = [-a_{ij}]_{mxn}$$

where all the entries of –M are negative of the respective entries of M.

### 4.1.7 BINARY OPERATIONS OVER MATRICES

To evolve matrices as a separate system of studies two binary operations of addition (+) and multiplication (·) are defined over them. These operations are not the binary operations of the field F over which these matrices are written but they are induced from the operations of F. In particular, we shall select entries of the matrices from the real field R unless stated otherwise.

### 4.1.8 ADDITIVE OPERATION

### 4.1.9 DEFINITION

Let $M = [a_{ij}]_{mxn}$ and $N = [b_{ij}]_{mxn}$ be two m x n matrices over the real field R. The addition M + N is defined as,

$$M + N = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

That is, the corresponding entries of the matrices M and N are added according to the additive operation of the field R.

For example,

$$\text{if } M = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

then

$$M + N = \begin{bmatrix} 1+2 & 2+1 & -3+3 \\ 5+1 & 0+0 & 2+2 \\ 1+1 & -1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 0 & 4 \\ 2 & -2 & 2 \end{bmatrix}$$

**CAUTION** The reader may note this point that the addition of two matrices is possible only if the numbers of rows and columns of M are the same as those in N.

The matrices, whose addition is possible to be made, are called matrices **conformable for addition.**

### 4.1.10 THEOREM (ADDITION)

For the matrices S, T and W conformable for addition

(i) $S + T = T + S$ (commutative property)

and (ii) $(S + T) + W = S + (T + W)$. (associative property)

**PROOF** Let $S = [a_{ij}]_{m\times n}$, $T = [b_{ij}]_{m\times n}$ and $W = [c_{ij}]_{m\times n}$.

$a_{ij}, b_{ij}, c_{ij} \in R$, when $1 \le i \le m$, $1 \le j \le n$.

Then $S + T = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$ (definition of addition)

$= [b_{ij} + a_{ij}]_{m\times n}$, $(a_{ij}, b_{ij} \in R)$.

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$ (definition)

$= T + S$.

It proves (i) and establishes the commutative property of matrices (conformable for addition) under the operation of addition.

For (ii),

$(S + T) + W = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ (Definition of addition)

$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$ (Definition of addition).

$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}$ (Additive Associativity of R).

$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$ (Definition)

$= S + (T + W)$.

It, therefore, establishes the associative property of matrices under (+).

### 4.1.11 EASY DEDUCTIONS

(i) Zero matrix $[0] = \mathbf{0}$ is conformable for addition to any m x n matrix.

i.e. $[0]_{m\times n} + [a_{ij}]_{m\times n} = [0 + a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} + [0]_{m\times n}$

So zero matrix behaves as the identity matrix under the operation of addition of matrices of any order.

(ii) For every matrix $M = [a_{ij}]_{m\times n}$, there exists a matrix $-M = [-a_{ij}]_{m\times n}$ which is conformable for addition to M,

such that $M + (-M) = [a_{ij}]_{m\times n} + [-a_{ij}]_{m\times n}$

$= [a_{ij} - a_{ij}]_{m\times n}$

$= [0]_{m\times n} = (-M) + M.$

The inverse property of matrices is observed under addition.

Thus;

### 4.1.12 THEOREM

The set of all m x n matrices over the real field R, forms an abelian group under the binary operation of matrix addition.

### 4.1.13 MULTIPLICATIVE OPERATION

The operation of multiplication attached with matrices is of two types.

(A) Scalar multiplication

and (B) Matrix multiplication

### 4:1.14 DEFINITION

For $\alpha \in R$, and the matrix $M = [a_{ij}]_{m\times n}$, the multiplication $\alpha \cdot M$ of M by $\alpha$ is defined to be equal to $[\alpha\, a_{ij}]_{m\times n}$

that is, $\alpha$ is multiplied (multiplication operation of R with M) with every entry of M.

For example, if

$$M = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 8 \\ 2 & 3 & 1 \end{bmatrix}$$ and $3 \in R$, then

$$3M = \begin{bmatrix} 3\cdot2 & 3\cdot3 & 3\cdot1 \\ 3(-1) & 3\cdot0 & 3\cdot8 \\ 3\cdot2 & 3\cdot3 & 3\cdot1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 3 \\ -3 & 0 & 24 \\ 6 & 9 & 3 \end{bmatrix}$$

**NOTE** (i) Scalar multiplication does not change the order of the matrices.

(ii) The scalar multiplication has a commutative property with matrices.

(iii) **Scalar Multiplication** is defined as the multiplication of the field elements with the matrices. Thus it is function from F x M into M.

### 4.1.15 THEOREM (SCALAR MULTIPLICATION)

Let $M = [a_{ij}]_{m\times n}$ and $N = [b_{ij}]_{m\times n}$ be two m x n matrices over the field R and $\alpha, \beta \in R$. Then ;

(i) $(\alpha + \beta)M = \alpha M + \beta M.$

(ii) $\alpha(M + N) = \alpha M + \alpha N$

(iii) $\alpha(\beta M) = (\alpha\beta)M.$

(iv) $1\cdot M = M,$ 1 is the identity of the field R.

**PROOF**

(i) $(\alpha + \beta)M = (\alpha + \beta)[a_{ij}]_{m\times n}, a_{ij}, \alpha, \beta \in R$

$= [(\alpha + \beta)\cdot a_{ij}]_{m\times n}$

$= [\alpha a_{ij} + \beta a_{ij}]_{m\times n}$ (Left distributive law of '·' over '+' in R).

$= [\alpha a_{ij}]_{m\times n} + [\beta a_{ij}]_{m\times n}$ (Addition of matrices)

$= \alpha[a_{ij}]_{m\times n} + \beta[a_{ij}]_{m\times n}$ (Scalar multiplication of matrices).

$= \alpha M + \beta M.$

(ii) $\alpha(M + N) = \alpha([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

$= \alpha([a_{ij} + b_{ij}]_{m\times n})$ (Addition of matrices)

$= [\alpha(a_{ij} + b_{ij})]_{m\times n}$ (Scalar multiplication)

$= [\alpha\, a_{ij} + \alpha\, b_{ij}]_{m\times n}$ (Distributive property of · over +in R)

$= [\alpha\, a_{ij}]_{m\times n} + [\alpha\, b_{ij}]_{m\times n}$ (Addition of matrices)

$= \alpha[a_{ij}]_{m\times n} + \alpha[b_{ij}]_{m\times n}$ (Scalar multiplication)

$= \alpha M + \alpha N.$

Thus the scalar multiplication is distributive over the addition of matrices.

(iii) $\alpha\cdot(\beta M) = \alpha\cdot(\beta[a_{ij}]_{m\times n}) = \alpha\cdot[\beta\, a_{ij}]_{m\times n}$

$= [\alpha\beta\, a_{ij}]_{m\times n}$, $\alpha, \beta, a_{ij} \in R.$

$= (\alpha\beta)\,[a_{ij}]_{m\times n}$

$= (\alpha\beta)M.$

The scalar multiplication of matrices is associative.

(iv) $1(M) = 1\cdot([a_{ij}]_{m\times n})$

$= [1\cdot a_{ij}]_{m\times n}$, $I_R = 1$(one)

$= [a_{ij}]_{m\times n}$, (definition of identity)

$= M.$

This completes the proof of the above theorem.

By the theorems [4.1.12 and 4.1.15] it follows that;

### 4.1.16 THEOREM

The set of all m x n matrices over the real field R together with the binary operations of matrix addition and scalar multiplication forms a vector space over the field R.

### 4.1.17 MATRIX MULTIPLICATION

As the addition of two matrices has been defined, the operation of multiplication is going to be introduced over the matrices. A little care is needed to realise that whether the matrices are conformable for multiplication or not. If we understand the rows and columns of a matrix as row and column vectors respectively then the product of two matrices S and T is taken to be the scalar product of row vectors of S with the column vectors of T. That is, ith row of S multiplied with jth column of T will produce the element for S·T (understood as ST) at the intersection of ith row and jth column. In this way the matrix S is conformable for multiplication with T if the number of columns of S is equal to the number of rows in T.

### 4.1.18 DEFINITION

For the matrices $M = [a_{ij}]_{m\times t}$ and $N = [b_{ij}]_{t\times n}$ over the field R, their product M·N is defined to be the matrix $[c_{ij}]_{m\times n}$, where

$$c_{ij} = [a_{i1}, a_{i2}, \ldots a_{it}] \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{tj} \end{bmatrix} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \cdots\cdots + a_{it}\, b_{tj}$$

i.e.

$$M \cdot N = [c_{ij}]_{m \times n}$$

It may be noted that the number of rows of M·N will be same as that of M and number of columns as N has.

For example, if

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 0 \end{bmatrix}_{2\times 3}, \quad \begin{bmatrix} 1 & -1 \\ 2 & 8 \\ 7 & 6 \end{bmatrix}_{3\times 2} = N$$

$$M \cdot N = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 7 & 1(-1) + 2\cdot 8 + 3\cdot 6 \\ 5\cdot 1 + 7\cdot 2 + 0\cdot 7 & 5(-1) + 7\cdot 8 + 0\cdot 6 \end{bmatrix} = \begin{bmatrix} 26 & 33 \\ 19 & 51 \end{bmatrix},$$

which is a 2 x 2 matrix.

**REMARKS**

1. The matrix multiplication could be remembered as "row x column" multiplication.
2. If M and N are m x n matrices then M · N is not a m x n matrix if $m \neq n$.
3. If M and N are square matrices of order $n^2$ then M · N is also square of the same order. Thus the binary operation of matrix multiplication over square matrices is closed.
4. For an arbitrary matrix A, $0.A = \mathbf{0} = A.0$

### 4.1.19 THEOREM (MULTIPLICATION)

Let $S = [a_{ij}]$, $T = [b_{ij}]$ and $W = [c_{ij}]$ be matrices over the field R and they are all conformable for addition and multiplication as indicated below, then,

(i) $S \cdot T = S \cdot W \not\Rightarrow T = W.$

(Cancellation law: left or right for multiplication is not observed)

(ii) $S \cdot T = \mathbf{0} \not\Rightarrow S = \mathbf{0}$ or $T = \mathbf{0}$

(iii) $S \cdot T \neq T \cdot S,$

(iv) $S\cdot(T\cdot W) = (S\cdot T)\cdot W$

(v) $S\cdot(T + W) = S\cdot T + S\cdot W$

(vi) $\cdot(S + T)\cdot W = S\cdot W + T\cdot W$

(vii) $\alpha\,(S\cdot T) = (\alpha S)\cdot T = S\cdot(\alpha T), \quad \alpha \in R.$

**PROOF**

(i) This statement can easily be supported by an example.

$$\text{If} \quad S = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, \; T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\text{and} \quad W = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

be all 3 x 3 matrices, then

$$S\cdot T = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix} \quad \text{and} \quad S\cdot W = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$$

Clearly $ST = SW$, but $T \neq W$ at all.

(ii) This statement can also be supported by an example.

$$\text{If} \quad S = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$ be two non-zero matrices and their product

$$S \cdot T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

So $S \cdot T = \mathbf{0}$ does not imply that either $T = \mathbf{0}$ or $S = \mathbf{0}$

(iii) Suppose that $S = [a_{ij}]$ and $T = [b_{ij}]$ are m x m matrices, then

$$S \cdot T = [c_{ij}]_{m \times m},\ c_{ij} = [a_{i1}, a_{i2}, \cdots a_{im}] \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj}$$

But $T \cdot S = [d_{ij}]_{m \times m}$, where $d_{ij} = [b_{i1}\ b_{i2} \cdots b_{im}] \cdot \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = b_{i1}a_{1j} + b_{i2}b_{2j} + \ldots + b_{im}a_{mj}$

Clearly $c_{ij}\ (1 \le i \le m,\quad 1 \le j \le m) \ne d_{ij}$

So in general, this property does not hold. Thus the **matrix multiplication is not commutative.**

For example, if

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 2 & 8 \\ 0 & 1 \end{bmatrix}$$

$$\text{then } ST = \begin{bmatrix} 2 & 8 \\ 4 & 17 \end{bmatrix} \ne \begin{bmatrix} 18 & 6 \\ 2 & 1 \end{bmatrix} = TS.$$

(iv) Let $S = [a_{ij}]$, $1 \le i \le m,\ 1 \le j \le n$

$T = [b_{jk}]$, $1 \le k \le t$

and $W = [w_{kl}]$, $1 \le l \le p.$

be $m \times n$, $n \times t$ and $t \times p$ matrices respectively. Then

$S.(TW) = S([b_{jk}] \cdot [w_{kl}])$

$= S[d_{jl}]$

where

$$[d_{jt}] = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \\ b_{j1}\, b_{j2} \cdots\cdots b_{jt} \\ \cdots\cdots \\ \cdots\cdots \end{bmatrix} \begin{bmatrix} \cdots\cdots w_{1t} \cdots\cdots \\ \cdots\cdots w_{2t} \cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots w_{tt} \cdots\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ \cdots\cdots d_{jt} \cdots\cdots \\ \vdots \end{bmatrix}$$

is $n \times p$ matrix

and $d_{jt} = b_{j1}w_{1t} + b_{j2}w_{2t} + \cdots\cdots + b_{jt} \cdot w_{tt}$ ——— (a)

Thus

$$S(TW) = S[d_{jt}] = [a_{ij}]\,[d_{jt}]$$

$$= [e_{it}]$$

$$= \begin{bmatrix} \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{i1}\ a_{i2} \cdots\cdots a_{in} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{bmatrix} \begin{bmatrix} \cdots\cdots d_{1t} \cdots\cdots \\ \cdots\cdots d_{2t} \cdots\cdots \\ \vdots \\ \vdots \\ \cdots\cdots d_{nt} \cdots\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ \cdots\cdots e_{it} \cdots\cdots \\ \vdots \\ \vdots \end{bmatrix}$$

is a $m \times p$ matrix.

where

$$e_{it} = a_{i1}d_{1t} + a_{i2}d_{2t} + \cdots\cdots + a_{in}d_{nt}$$ ——— (b)

By substituting $j = 1, 2, \ldots n$, we get $d_{1t}, d_{2t}, \ldots\ldots, d_{nt}$ from (a)

By substituting these values in (b) we get,

$$e_{it} = a_{i1}(b_{11}w_{1t} + b_{12}w_{2t} + \cdots\cdots + b_{1t}w_{tt})$$
$$+ a_{i2}(b_{21}w_{1t} + b_{22}w_{2t} + \cdots\cdots b_{2t}w_{tt})$$
$$+ \cdots\cdots + a_{in}(b_{n1}w_{1t} + b_{n2}w_{2t} + \cdots\cdots + b_{nt}w_{tt})$$ ——— (c)

For

$$(ST)W = ([a_{ij}]\,[b_{jk}])\,W$$

$$= [f_{ik}]\,[w_{kl}]$$

where

$$[f_{ik}] = \begin{bmatrix} \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ a_{i1}a_{i2} \cdots\cdots a_{in} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{bmatrix} \begin{bmatrix} \cdots\cdots b_{1k} \cdots\cdots \\ \cdots\cdots b_{2k} \cdots\cdots \\ \vdots \\ \cdots\cdots b_{nk} \cdots\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ \cdots\cdots f_{ik} \cdots\cdots \\ \vdots \end{bmatrix}$$

is a $m \times t$ matrix

and $f_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots\cdots + a_{in}b_{nk}$ ——— (d)

Thus

$$(ST)W = [f_{ik}]\,[w_{kl}]$$

$$= [g_{il}] = \begin{bmatrix} \vdots \\ \cdots\cdots g_{il} \cdots\cdots \\ \vdots \end{bmatrix}$$

where $g_{il} = f_{i1}w_{1l} + f_{i2}w_{2l} + \cdots\cdots + f_{it}w_{tl}$ ——— (e)

and $(ST)W$ is $m \times p$ matrix

From (d), after getting $f_{i1}, f_{i2}, \ldots, f_{it}$ and putting in (e), we get that

$$\begin{aligned} g_{il} &= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots\cdots + a_{in}b_{n1})\,w_{1l} \\ &\quad + (a_{i1}b_{12} + a_{i2}b_{22} + \cdots\cdots + a_{in}b_{n2})\,w_{2l} \\ &\quad + \cdots\cdots + (a_{i1}b_{1t} + a_{i2}b_{2t} + \cdots\cdots + a_{in}b_{nt})w_{tl} \\ &= a_{i1}\,(b_{11}w_{1l} + b_{12}w_{2l} + \cdots\cdots + b_{1t}w_{tl}) + \ldots \\ &\quad \ldots + a_{in}\,(b_{n1}w_{1l} + b_{n2}w_{2l} + \cdots\cdots + b_{nt}w_{tl}) \\ &= e_{il}, \text{ for each } i \text{ and } l. \text{ It thus proves (iv).} \end{aligned}$$

(v) Let $S = [a_{ij}]$, $[b_{jk}] = T$ and $W = [c_{jk}]$ be $m \times n$, $n \times p$ and $n \times p$ matrices respectively where, $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$.

Then

$$\begin{aligned} S(T + W) &= [a_{ij}]\,([b_{jk}] + [c_{jk}]) \\ &= [a_{ij}]\,[b_{jk} + c_{jk}]. \\ &= [t_{ik}] \end{aligned}$$

$$= \begin{bmatrix} a_{i1} a_{i2} \ldots\ldots a_{in} \end{bmatrix} \begin{bmatrix} \ldots\ldots\ b_{1k} + c_{1k} \ldots\ldots \\ \ldots\ldots\ b_{2k} + c_{2k} \ldots\ldots \\ \\ \ldots\ldots\ b_{nk} + c_{nk} \ldots\ldots \end{bmatrix}$$

where

$$t_{ik} = a_{i1}(b_{1k} + c_{1k}) + a_{i2}(b_{2k} + c_{2k}) + \ldots\ldots + a_{in}(b_{nk} + c_{nk})$$

For $ST + SW = [a_{ij}][b_{jk}] + [a_{ij}][c_{jk}]$

$= [\ell_{ik}] + [f_{ik}]$

$= [\ell_{ik} + f_{ik}]$

where $\ell_{ik} + f_{ik} = (a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots\ldots + a_{in}b_{nk})$

$+ (a_{i1}c_{1k} + a_{i2}c_{2k} + \ldots\ldots + a_{in}k_{nk})$

$= a_{i1}(b_{1k} + c_{1k}) + a_{i2}(b_{2k} + c_{2k}) \ldots\ldots$

$+ \ldots\ldots + a_{in}(b_{nk} + c_{nk})$

$= t_{ik}$, for each i, k.

It thus proves the equality.

This equality explains the right distributive property of multiplication over the matrix addition.

(vi) Let $S = [a_{ij}]$, $T = [b_{ij}]$

and $W = [c_{jk}]$

be matrices of order $m \times n$, $m \times n$ and $n \times p$ respectively.

Then,

$(S + T)\cdot W = ([a_{ij}] + [b_{ij}])[c_{jk}]$

$= [a_{ij} + b_{ij}][c_{jk}]$

$= [a_{ij}\, c_{jk} + b_{ij}c_{jk}]$ from (v)

$= [a_{ij}\, c_{jk}] + [b_{ij}c_{jk}]$

$= [a_{ij}][c_{jk}] + [b_{ij}][c_{jk}]$ using (v)

$= SW + TW.$

This equality explains the left distributive property of multiplication over the matrix addition.

(vii) Let $S = [a_{ij}]$ and $T = [b_{jk}]$ where

$1 \le m \le i$, $1 \le n \le j$, and $1 \le k \le p$

$\alpha \cdot (ST) = \alpha \cdot ([a_{ij}]\,[b_{jk}]) = \alpha\,[d_{ik}] = [\alpha d_{ik}]$

where $d_{ik} = a_{i1}\,b_{1k} + a_{i2}b_{2k} + \ldots\ldots + a_{in} \cdot b_{nk}$

$\alpha\, d_{ik} = \alpha(a_{i1}\,b_{1k} + a_{i2}b_{2k} + \ldots\ldots + a_{in} \cdot b_{nk})$

$= (\alpha a_{i1})\,b_{1k} + (\alpha a_{i2})\,b_{2k} + \ldots\ldots + (\alpha a_{in})\,b_{nk}$ ..... (A)

$[\alpha\, d_{ik}] = [\alpha\, a_{ij}]\,[b_{nk}]$

$= (\alpha\, S)\,T.$

From (A) $\alpha\, d_{ik} = a_{i1}(\alpha b_{1k}) + a_{i2}(\alpha b_{2k}) + \ldots\ldots + a_{in}\,(\alpha b_{nk})$

$[\alpha\, d_{ik}] = [a_{ij}]\,[\alpha b_{jk}]$

$= S.\,(\alpha T)$

Hence the equality $\alpha(ST) = (\alpha S)T = S(\alpha T)$ is proved.

The properties of matrix multiplication established so for lead us to the following theorem:

### 4.1.20 THEOREM

The set of all $n \times n$ square non-singular matrices over a field F forms a non-abelian group under the binary operation of matrix multiplication.

### 4.1.21 DIFFERENT KINDS OF SQUARE MATRICES

As for the multiplicative structure of matrices is concerned, square matrices play an important role. Some of the important square matrices are defined.

### 4.1.22 DEFINITIONS

A square matrix $M = [a_{ij}]$ of order $m \times m$ is called ;

1. **Symmetric**, if $M = [a_{ij}] = [a_{ji}] = M^t$, the transpose of M.

For example, $S = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & -1 \end{bmatrix}$ is a $3 \times 3$ symmetric matrix.

2. **Skew-symmetric** if $M^t = [a_{ji}] = [-a_{ij}] = -M$

For example, $K = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$ is skew–symmetric of order 3x3

3. **Hermitian,** if transpose of conjugate of M is equal to M.

i.e. M is Hermitian if and only if

$(\bar{M})^t = [\bar{a_{ji}}] = [a_{ij}] = M.$

For example, $H = \begin{bmatrix} 1 & 1+i & i \\ 1-i & 2 & 2-i \\ -i & 2+i & -1 \end{bmatrix}$ is a Hermitian 3 x 3 matrix

4. **Skew-Hermitian,** if $(\bar{M})^t = -M$.

or $(\bar{M})^t + M = \mathbf{0}$

For example, $K = \begin{bmatrix} 0 & 1+i & i \\ -1+i & 0 & 2-i \\ i & -2-i & 0 \end{bmatrix}$ is skew–Hermitian 3x3 matrix

5. **Diagonal,** if $a_{ij} = 0$, for $i \neq j$ and atleast one $a_{ii} \neq 0$.

For example, $D = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & v \end{bmatrix}$ is a diagonal real 3x3 matrix if not all $\lambda$, $w$, $v \in R$ be zero simultaneously and not all essentially be different.

If $0 \neq \lambda = w = v$ then D is called **scalar** matrix. If $\lambda = w = v = 1$ then D is called **unit** or multiplicative identity matrix.

6. **Upper triangular,** if $a_{ij} = 0$, for $i > j$ and there exists atleast one non-zero entry for $i < j$.

For example, $T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is upper triangular 3x3 matrix

7. **Lower triangular,** if $a_{ij} = 0$, for $i < j$ and there exists atleast one non-zero. entry for $i > j$.

For example, $L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is lower triangular matrix.

If the diagonal elements are all zero the matrix is called strictly upper / lower. triangular.

8. **Inverse** of $N = [b_{ij}]$ is M if $M.N = I = N.M$, where I is the $3 \times 3$ identity matrix. The inverse M of N is written as $N^{-1}$. If a square matrix M has its inverse then M is called **non-singular** or invertable otherwise **singular.**

Note that, a square diagonal matrix is singular if any of the diagonal entry is zero, otherwise it is non - singular.

### 4.1.23 THEOREM

Let M, N be two matrices conformable for their respective operations of addition and multiplication. Then,

(i) $(M^t)^t = M$

(ii) $(M + N)^t = M^t + N^t$

(iii) $(M \cdot N)^t = N^t \cdot M^t$

(iv) $M + M^t$ is symmetric and $M - M^t$ is skew-symmetric.

### PROOF

(i) If $M = [a_{ij}]$ then $M^t = [a_{ji}]$ and also

$(M^t)^t = [a_{ji}]^t = [a_{ij}] = M.$

(ii) If $M = [a_{ij}]$ and $N = [b_{ij}]$ then,

$(M+N)^t = [a_{ij} + b_{ij}]^t = [c_{ij}]^t = [c_{ji}] = [a_{ji} + b_{ji}]$

$= [a_{ji}] + [b_{ji}] = [a_{ij}]^t + [b_{ij}]^t = M^t + N^t$

(iii) $(M \cdot N)^t = ([a_{ij}] \cdot [b_{jk}])^t$

$= [c_{ik}]^t = [c_{ki}] = [b_{kj}] \cdot [a_{ji}] = N^t \cdot M^t$

$$\text{where } c_{ik} = [a_{i1}\ a_{i2} \cdots a_{im}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{bmatrix}, \quad \text{when } 1 \le j \le m$$

$$= a_{i1}\ b_{1k} + a_{i2}\ b_{2k} + \cdots\cdots + a_{im}\ b_{mk}$$

$$\text{and } c_{ki} = b_{k1}\ a_{1i} + b_{k2} \cdot a_{2i} + \cdots\cdots + b_{km} \cdot a_{mi}$$

(iv) Consider $(M + M^t)^t = M^t + (M^t)^t$, by (ii)

$= M^t + M$, by (i)

$= M + M^t$, (Commutative property).

Thus $M + M^t$ is a symmetric matrix.

Further $(M - M^t)^t = M^t - (M^t)^t$, by (ii)

$= M^t - M$, by (i)

$= -(M - M^t)$

Hence $M - M^t$ is skew-symmetric.

**COR.1.** Since $2A = (A + A^t) + (A - A^t)$, therefore each square matrix A can be written as sum of a symmetric and a skew-symmetric matrix.

**COR.2** If $M = [a_{ij}]$ is skew–symmetric

then, $[a_{ij}]^t = -[a_{ij}]$

$\Rightarrow [a_{ji}] + [a_{ij}] = \mathbf{0}$

$\Rightarrow [a_{ji} + a_{ij}] = \mathbf{0}$

$\Rightarrow [a_{ii} + a_{ii}] = \mathbf{0}$, if $i = j$

$\Rightarrow [2a_{ii}] = \mathbf{0} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0 \quad \forall_i$

Hence all the diagonal entries of a skew - symmetric matrix are zero. If $i \neq j$ then $a_{ji} = -a_{ij}$.

## 4.1.24 THEOREM

Let S be an inverse of a matrix T. Then S is unique such matrix

**PROOF** If S is inverse of T then, $S \cdot T = I = T \cdot S$ ..... (i)

If S is not unique then, let M be another such matrix which is inverse of T. Then,

$$M \cdot T = I = T \cdot M \quad \text{and} \quad \ldots\ldots \text{(ii)}$$

$$M = IM = (ST)M, \text{ by (i)}$$
$$= S(TM), \text{ associativity}$$
$$= SI = S, \text{ by (ii)}$$

which provides us contradiction with that M is different from S.

Hence S is unique inverse of T.

### 4.1.25 THEOREM

Let S and T be two complex square matrices conformable for their respective operations and $\overline{S}$ and $\overline{T}$ be their respective conjugates. Then,

(i) $\overline{(\overline{S})} = S$

(ii) $\overline{(S + T)} = \overline{S} + \overline{T}$

(iii) $\overline{(ST)} = \overline{S} \cdot \overline{T}$

(iv) $(\overline{S})^t = \overline{(S^t)}$

(v) $T + (\overline{T})^t$ is Hermitian and $T - (\overline{T})^t$ is skew-Hermitian.

**PROOF**

(i) If $S = [a_{ij}]$, $a_{ij} \in \mathbb{C}$, then

$$\overline{S} = [\overline{a_{ij}}] \text{ and } \overline{(\overline{S})} = [\overline{\overline{a_{ij}}}] = [a_{ij}] = S$$

(ii) If $S = [a_{ij}]$ and $T = [b_{ij}]$, then

$$\overline{(S + T)} = [\overline{a_{ij} + b_{ij}}] = [\overline{a_{ij}} + \overline{b_{ij}}]$$
$$= [\overline{a_{ij}}] + [\overline{b_{ij}}] = \overline{S} + \overline{T}.$$

(iii) $$\overline{(S \cdot T)} = \overline{[a_{ij}] \cdot [b_{jk}]} = [\overline{a_{i1} \cdot b_{1k} + a_{i2} \cdot b_{2k} + \cdots + a_{im} \cdot b_{mk}}]$$
$$= [\overline{a_{i1}} \cdot \overline{b_{1k}} + \overline{a_{i2}} \cdot \overline{b_{2k}} + \cdots \overline{a_{im}} \cdot \overline{b_{mk}}]$$
$$= [\overline{a_{ij}}] \cdot [\overline{b_{jk}}] = \overline{S} \cdot \overline{T}.$$

(iv) $(\overline{S})^t = [\overline{a_{ij}}]^t = [\overline{a_{ji}}] = [\overline{a_{ji}}] = (\overline{S^t})$

(v) Let $M = T + (\overline{T})^t$. Then

$$(\overline{M})^t = \overline{(T + (\overline{T})^t)^t} = (\overline{T} + (\overline{\overline{T}})^t)^t, \text{ by (ii), (iv)}$$

$$= (\overline{T} + (\overline{\overline{T}})^t)^t, \text{ by (iv)}$$

$$= (\overline{T} + (T)^t)^t, \text{ by (i)}$$

$$= (\overline{T})^t + (T^t)^t, \text{ by 4.1.23 (ii)}$$

$$= (\overline{T})^t + T, \text{ by theorem 4.1.23 (i).}$$

$$= T + (\overline{T})^t, \text{ (commutative addition)}$$

$$= M$$

Thus M is a Hermitian matrix. Similarly,

If $N = T - (\overline{T})^t$, then

$$(\overline{N})^t = \overline{(T + (\overline{\overline{T}})^t)^t}$$

$$= (\overline{T} - (\overline{\overline{T}})^t)^t$$

$$= (\overline{T} - (T)^t)^t$$

$$= (\overline{T})^t - (T^t)^t$$

$$= (\overline{T})^t - T$$

$$= -(T - (\overline{T})^t)$$

$$= -N,$$

which proves that $T - (\overline{T})^t$ is skew-Hermitian.

**COR.1** Every square matrix A over the complex field $\mathbb{C}$ is written as sum of a Hermitian and a skew–Hermitian matrix.

As, $2A = A - (\overline{A})^t + A + (\overline{A})^t$

and $A = \frac{1}{2}(A - (\overline{A})^t) + \frac{1}{2}(A + (\overline{A})^t)$

**COR.2** If $A = [a_{ij}]$ is skew - Hermitian, then

$(\overline{A})^t = -A$

$\Rightarrow [\overline{a_{ji}}] = -[a_{ij}]$

$\Rightarrow [\overline{a_{ji}} + a_{ij}] = \mathbf{0}$

If $j = i$, then

$[\overline{a_{ii}} + a_{ii}] = \mathbf{0}$

$\Rightarrow$ real parts of $a_{ii}$ are zero or $a_{ii} = 0$

Hence the diagonal entries of A are either zero or imaginary. If $j \neq i$, then $\bar{a}_{ji} = -a_{ij}$.

**NOTE** The name, Hermitian and skew-Hermitian, associated with a matrix is after the name of a very famous French mathematician Hermite (1822 — 1901).

### 4.1.26 DEFINITION

A square matrix A over the real field R is called **orthogonal,** if $A^t \cdot A = I = A \cdot A^t$.

For example, $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

is an orthagonal matrix.

That is, $A^t$ is the inverse of A or $A^t = A^{-1}$.

From the definition it follows that;

(i) A is orthogonal if and only if $A^{-1}$ is orthogonal.

(ii) If A is orthogonal then A is symmetric if and only if $A^2 = I$

(iii) If A and B are orthogonal then AB and BA are also orthogonal.

(iv) An orthogonal matrix is also understood as a square matrix whose column vectors are of unit length and their dot products are zero.

### 4.1.27 DEFINITION

A square matrix A with entries from the complex field $\mathbb{C}$ is called **unitary**, if

$$(\bar{A})^t \cdot A = I = A \cdot (\bar{A})^t.$$

That is, $(\bar{A})^t$ is inverse of the matrix A.

or $(\bar{A})^t = A^{-1}$.

For example, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. is a unitary matrix.

It thus follows from the definition that,

(i) A is unitary if and only if $\bar{A}$ is unitary.

(ii) If A, B are unitary then AB, BA are also unitary.

(iii) If A is unitary, then $A^2 = I$ if and only if A is Hermitian.

(iv) If A is Hermitian, then $A^2 = I$ if and only if A is unitary.

(v) A unitary matrix U is remembered as a square matrix whose column vectors have length $c_i$ . $c_i = 1$ and their dot product is zero i.e $c_i$ . $c_j = 0, j \neq i$.

### 4.1.28 DEFINITION

A square matrix A is called **periodic** if $A^{k+1} = A$, for a positive integer k. If k is such least integer then k is called the **period** of A.

For example, $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is a

periodic matrix of period 2.

Noted that,

$$A = A^{k+1} = A^k . A = A . A^k.$$

If A is non-singular, then

$A^k = I$, the identity matrix

If $k = 1$, then $A^2 = A$. Thus the matrices of period one are called **idempotents.**

A is **involutory** if $A^2 = I$, identity. involutory matrices are of period 2.

For example,

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ are involutory matrices.

### 4.1.29 DEFINITIQN

For a positive integer n, a square matrix P for which $P^n = \mathbf{0}$ is called **nilpotent.** If n is the least such integer then n is called the nilpotency index of P or P is called nilpotent of nilpotency class n.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is a nilpotent matrix of nilpotency

class 2.

## EXERCISE 4.1

**1.** Find a $2 \times 2$ matrix B such that (i) $2AB = C + 3D$

(ii) $2A + B = 3CD$, where

$$A = \begin{bmatrix} 5 & -4 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 21 & -18 \\ -17 & 16 \end{bmatrix} \text{ and } \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**2.** (i) Let $A = [1 \;\; 2 \;\; -1]$ and $B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Find AB and BA. Examine whether $AB = BA$?

(ii) If $\theta = \phi + \frac{5\pi}{2}$ then show that $A.B = \mathbf{0}$, where

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos^2\varphi & \cos\varphi \sin\varphi \\ \cos\varphi \sin\varphi & \sin^2\phi \end{bmatrix}.$$

*(iii)* *(a)* If A and B are $3 \times 3$ real matrices such that $\det(AB^2) = 18$, $\det(A^2B) = 12$, then find $\det(3A)$, $\det(B^{-1})$.

*(b)* If A and B are $4 \times 4$ matrices such that $\det(A^3B^4) = -27$, $\det(A^2B^3) = 9$, then find $\det(3B)$, $\det(A^{-1})$.

*(c)* If A and B are $6 \times 6$ matrices such that $\det(AB^2) = 72$, $\det(A^2B^2) = 144$, thus find $\det(AB^{-})$.

**3.** Find the real x if $[x \;\; x \;\; 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix} = 0$

**4.** Show that $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is a solution of the equation $A^3 - 4A^2 - 3A + 11\,I = 0$.

**5.** If $D = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then

(i) Compute $D^2$, $D^3$ and $D^4$.

(ii) Compute $D^n$ and describe a relation ship between $D^n$ and D for an arbitrary diagonal matrix.

**6.** Give examples of matrices A, B, C for which the following identities fail.

(i) If $A.B = 0$, then either $A = 0$ or $B = 0$

(ii) If $AB = AC$, and $A \neq 0$, then $B = C$

(iii) $(A + B)^2 = A^2 + 2AB + B^2$

(iv) $A^2 - B^2 = (A - B)(A + B)$.

(v) If $A^2 = I$, then $A = I$.

**7.** For any square matrix A, show that

(i) $A^tA$ and $AA^t$ are symmetric. (ii) $\frac{1}{2}(A + A^t)$ is symmetric.

(iii) $\frac{1}{2}(A - A^t)$ is skew-symmetric.

**8.** For a square symmetric matrix A, show that,

$P^tAP$ and $PAP^t$ are symmetric matrices, when P is square matrix of the same order as that of A.

**9.** For a square skew-symmetric matrix A, show that

$P^tAP$ and $PAP^t$ are skew-symmetric, where P is a square matrix of the same order as that of A.

**10.** Show whether the following matrices

(i) $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

are nilpotent? If they are nilpotent then find their nilpotency index.

**11.** Show, whether the matrix $\begin{bmatrix} 1 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent or periodic? If it is

periodic then find its period.

**12.** Show that the following matrices are idempotent;

(i) $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

**13.** Show that the following matrices are involutory;

(i) $\begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$.

**14.** Prove that $(\alpha A^t) = \alpha A^t$, for any real number $\alpha$.

**15.** If A and B are two n x n upper triangular (lower triangular) matrices, then show that AB is upper triangular (lower triangular). Deduce that $A^2$, $A^3$, ....., $A^n$ are upper (lower) triangular if A is upper (lower) triangular.

**16.** Show that the following matrices are non-singular :

(i) $\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$.

**17.** If A is an $n \times n$ matrix, then prove that

(i) If $(I - A)(I + A) = \mathbf{0}$, then $A = A^{-1}$, if A is non-singular

(ii) If $2A^2 - 6A + 2I = \mathbf{0}$, then $A^{-1} = 3I - A$.

(iii) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ then show that $(A - I)(A - 4I) = \mathbf{0}$. Hence evaluate $A^3 - A$.

**18.** If A, B, C are three non–singular n x n matrices, show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

**19.** If AB = BA for square matrices A and B, then prove that;

(i) $A^{-1}B^{-1} = B^{-1}A^{-1}$ (ii) $A^tB^t = B^tA^t$

(iii) $(\bar{A})^t(\bar{B})^t = (\bar{B})^t(\bar{A})^t$.

**20.** For a square matrix A, show that

(i) $A(\bar{A})^t$ is Hermitian

(ii) $(\bar{A})^tA$ is Hermitian

(iii) Express $M = \begin{bmatrix} 4+3i & 0 & 3i \\ 1 & i & 6 \\ 1-i & -1+i & 7 \end{bmatrix}$ as a sum matrix of a Hermitian matrix and a skew-Hermitian matrix.

**21.** Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $|A| \neq 0$, $A \neq I$. Then, find $a$, b, c, if

(i) A is idempotent

(ii) A is involutory.

**22.** Find two $2 \times 2$ matrices A and B for which

$AB = \mathbf{0}$ and $BA \neq \mathbf{0}$

**23.** Let A be a square matrix. Then show that,

(i) $(A^p)^t = (A^t)^p$, where p is any positive integer.

(ii) $(A^p)^{-1} = (A^{-1})^p$

(iii) $(A^{-1})^{-1} = A$

(iv) $(kA)^{-1} = \frac{1}{k}A^{-1}$, k being a non-zero scalar.

(v) $(A^{-1})^t = (A^t)^{-1}$

(vi) $(\bar{A})^{-1} = \overline{(A^{-1})}$

(vii) $(\overline{A^t})^{-1} = (\overline{A^{-1}})^t$

**24.** Determine the values of $a$, b, c if the matrix $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

**25.** Prove that A is orthogonal if and only if $A^{-1}$ is orthogonal.

**26.** Show that if A and B are orthogonal then AB, BA are orthogonal.

**27.** If $A = [a_{ij}]$ is a $m \times n$ matrix with complex entries then show that $A + \bar{A}$ is a matrix with real entries.

**28.** (i) If A and B are Hermitian matrices of the same order then prove that A + B is Hermitian.

(ii) If A and B are skew-Hermitian of the same order then show that A + B is skew–Hermitian.

**29.** If U is a unitary matrix, then prove that,

(i) $U^k$ is unitary for every positive integer k.

(ii) $U^2 = I$, if and only of U is Hermitian.

**30.** Express the following as sum of a symmetric matrix and a skew-symmetric matrix;

(i) $\begin{bmatrix} 3 & -2 & 1 \\ 5 & 2 & 3 \\ -1 & 6 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 12 & 4 & -5 \\ -3 & 5 & 6 \end{bmatrix}$, (iii) $\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$.

**31.** Verify the following equalities;

(i) $(AB)^t = B^tA^t$, for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 5 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

(ii) $(\overline{PQ})^t = \bar{Q}^t\bar{P}^t$, for $P = \begin{bmatrix} 2+i & 7-i \\ 3 & -i \\ 0 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 1+i & 7 \\ 2-i & -4 \end{bmatrix}$

(vii) $(\overline{M})^t = (\overline{M^t})$. for $M = \begin{bmatrix} 2+3i & 4-i & 2-i \\ 6+7i & 3+2i & 3-4i \\ 5 & 7+11i & 3 \end{bmatrix}$.

**32.** Find the values of $a$ and b if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2b \\ 5 & -3 & 1 \end{bmatrix}$

**33.** For the pair of matrices

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

verify that $(AB)^{-1} = B^{-1}.A^{-1}$.

**34.** Show that,

(i) Every Hermitian matrix can be expressed as A + iB, where A is real symmetric and B is real skew–symmetric.

(ii) Every skew–Hermitian matrix can be written as A +iB, where A is real skew-symmetric and B is real symmetric.

**35.** (i) If A is a matrix over the real field R, then $AA^t = \mathbf{0} \Rightarrow A = \mathbf{0}$.

(ii) If A is a matrix over the complex field $\mathbb{C}$, then $A\bar{A}^t = \mathbf{0} \Rightarrow A = \mathbf{0}$.

**36.** Prove that;

(i) If A is a Hermitian matrix then iA and –iA are skew–Hermitian.

(ii) If A is skew-Hermitian then iA and –iA are Hermitian.

**37.** Prove that every square matrix can be expressed as a sum P + iQ, where P and Q are Hermitian matrices.

**38.** For a skew-symmetric matrix A, prove that

$(A^n)^t = (-1)^n A^n$, for any +ve integer n.

Deduce that, if n is even then $A^n$ is symmetric and if n is odd then $A^n$ is skew-symmetric.

## 4.2. ELEMENTARY OPERATIONS ON MATRICES

Let A be an $m \times n$ matrix over a field F. Then certain elementary operations are defined over A by means of which a new matrix from the old is obtained. It involves two kinds of such operations (i) Row operations (ii) column operations.

### 4.2.1 ELEMENTARY ROW OPERATIONS

The operations over A are called **elementary row operations** of type,

I, if any two rows of A are interchanged

II, if any row of A is multiplied by a non-zero scalar of the field.

III, if any row of A is added to a scalar multiple of another row of A.

One may be aware that an operation,

"Multiplying second row by 2 and adding 3 times the second row to the third row", over A is not elementary row operation but consists of the successive application of elementary row operations of type II and type III.

Similarly the **elementary column operations** are defined over the columns of the matrix A. Structure wise there is no difference between the application of elementary row operation or column operation of the matrices. We shall concentrate on the elementary row operations in most of the cases.

**NOTATIONS:** i) Interchanging of ith and jth row shall be denoted by $R_{ij}$.

ii) k multiple of the ith row by $kR_i$

and iii) k th multiple of ith row, added to the jth row will be denoted by $kR_i + R_j$.

### 4.2.2 DEFINITION

A matrix B obtained, after successive application of finite number of elementary row operations on A in a definite order, is called **row equivalent** to A and is denoted by

$$R(A) = B. \text{ or } A \overset{R}{\sim} B.$$

Similarly if B is column equivalent to A then

$$B = C(A) \text{ or } A \overset{C}{\sim} B.$$

### 4.2.3 THEOREM

The relation of "row equivalent" over the matrices is reflexive, symmetric and transitive. Hence is an equivalence relation and all the row equivalent matrices form one **equivalence class.**

### 4.2.4 DEFINITION

A matrix $A = [a_{ij}]_{m,n}$ is said to be in its **row echelon** form if

(1) All the zero rows are below the non zero rows of A.

(2) The first non–zero entry in each row of A is further to the right than the first non–zero entries in all preceeding rows

For example,

The matrices,

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $[0, -1, 2]$ are in row echelon form, while the matrices

$$\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not in its row echelon form.

The first non-zero element in each row of a matrix in its row echelon form is called **pivot element** of that row.

### 4.2.5 DEFINITION

A matrix A is called in its **reduced row echelon form** if,

(i) A is in its row echelon form.

(ii) Pivot element of each row is 1.

and (iii) Each entry in the pivot column is zero except the pivot element 1.

For example, the matrices

$$[1, 2, 3], \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \text{ are in reduced echelon form}$$

It is important to note that ;

(i) Each non–zero row of a row echelon matrix contains exactly one pivot element.

(ii) Each column of an echelon matrix contains at most one pivot. A column of an echelon matrix that contains a pivot is called a **pivot column.**

(iii) The number of non–zero rows of an echelon matrix is equal to the number of pivot columns.

(iv) Every zero matrix is already in its echelon form.

(v) Some authors include the condition of obtaining pivot elements to be 1 in the row echelon form of a matrix instead in its reduced echelon form.

### 4.2.6 THEOREM

Every matrix $[a_{ij}]_{m\times n}$ can be transformed into its row reduced echelon form by means of successive application of a finite number of elementary row operations.

**PROOF** We assume that the given matrix A is non-zero. Consider the left most non-zero column of A, say, the jth column. If $a_{1j} = 1$, then we are already done. If $a_{1j} \neq 1$ then find least i such that $a_{ij} = 1$, for each i, then divide the ith row by $a_{ij}$ to get the pivot 1 and interchange $R_i$ with $R_1$. We obtain the matrix B this way. The binary operations $R_i - a_{ij}R_1$, for each $i \geq 2$ produce all the series in the jth column except the pivot 1 in $C_j$.

Next, we look for a column having a non–zero entry below the first row. If there is no such column then we have already reduced the matrix. If there is such a column, choose the left most such column, say, kth column where $k > j$. The column, preceding $C_k$ have all zeros below the first row. If the element $a_{2k} = 0$, we make it non–zero by choosing a non–zero entry $a_{hk}$,, where $h \neq 1$. If $a_{hk} = 1$ then perform the binary operation of interchanging $R_2$ and $R_h$. If $a_{hk} \neq 1$ then divide the row $R_h$ by $a_{hk}$ and get pivot of kth column as 1, at position (2, k). To find all the zeros in the kth column of A, apply the binary operation $a_{ik} - a_{ik}\, R_2 (i > 2)$ for each i.

There operations do not affect the columns preceding $C_k$. By continuing this process we transform A into its reduced echelon form. Since the matrix A has only finitely many rows, the process ends after finitely many steps.

**EXAMPLE** Transform the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$ into row reduced echelon form.

Since $a_{11} = 1$ and $a_{21} = 0$, therefore, the operations $R_3 - 2R_1$ and $R_4 - 4R_1$ produce zeros in $C_1$ except $a_{11} = 1$. Hence

$$R(A) \sim B = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

There is no 1 in $C_2$, therefore we perform the elementary row operation $\frac{1}{11} R_2$ to get the entry (2, 2) as 1. Then

$$R(B) \sim C = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

$$\text{Now } R(C) \sim C^* = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by, $R_3 + 11R_2$ and $R_4 + 11R_2$

To produce $a_{12} = 0$, we transform by $R_1 - 3R_2$ and get

$$R(C^*) \sim B^* = \begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the desired row reduced echelon form of A.

### 4.2.7 THEOREM

Let A be an n x n upper triangular matrix. If all the diagonal entries of A are non–zero (that is, $a_{ii} \neq 0$, for all i), then A can be transformed to the identity matrix $I_n$.

**PROOF** Take an upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ :::: & :::: & :::: & :::: & :::: \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix}$$

Multiply the last row by $\frac{1}{a_{nn}}$ and get a entry 1 at (n, n) place of the transformed matrix.

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & a_{1n} \\ 0 & a_{22} & a_{23} & : & a_{2n} \\ :::: & :::: & :::: & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

Operate over B by the operations,

$$R_1 - a_{1n}R_n, \quad R_2 - a_{2n}R_n \ \ldots\ldots \text{ and } \quad R_{n-1} - a_{n-1,n}R_n.$$

when the nth column has all the zeros except 1 at (n, n)th place. This matrix is transformed then to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1,n-1} & 0 \\ 0 & a_{22} & a_{23} & \ldots & a_{2,n-1} & 0 \\ :::: & :::: & :::: & :::: & a_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

Now this matrix has $n - 1 \times n - 1$ upper triangular sub-matrix. We may repeat the process on the first $(n - 1)$ rows and columns to change $a_{n-1,\,n-1} = 1$ and change every preceding entry in the $(n - 1)$st column to a zero. Continuing in this way, we arrive at the identity matrix $I_n$.

## 4.2.8 THEOREM

An n x n matrix A is non–singular if and only if it can be transformed into the identity matrix by the elementary row operations.

**PROOF** An upper triangular matrix is non–singular if all the diagonal entries are non–zero, which can be transformed to the identity, by the theorem [4.2.7]. If A can be transformed into the identity matrix then it is a special case of the triangular matrix, all of whose diagonal entries are non–zero.

## 4.2.9 DEFINITION

The matrix E obtained, by operating any of the single elementary row operation to the identity matrix, is called the elementary matrix.

For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ corresponds to the elementary row operation $kR_2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ corresponds to the elementary row operation $R_{23}$ of interchanging $R_2$ and $R_3$.

$$\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ corresponds to the elementary row operation of $R_1 + kR_2$

are the elementary matrices row equivalent to the identity matrix $I_3$. These three are elementary matrices which are square and non–singular and so is their product.

### 4.2.10 OBSERVATIONS

1. An elementary operation on a square matrix A amounts to pre-multiplying A by its corresponding elementary matrix.

2. If $R(A) \sim B$, then $B = EA$ where E is a non–singular matrix obtained as the product of elementary matrices taken in definite order, corresponding to the application of the elementary row operation on A.

3. A matrix A is non-singular if it can be written as a product of elementary matrices.

4. $R(A) \sim B$ iff $B = EA$,

5. The reduced echelon form of any non–singular matrix is the identity matrix.

### 4.2.11 THEOREM

The proof for the singular case is beyond the scope of this book.

The reduced echelon form of a matrix A is unique.

**PROOF** If A is non-singular then it reduces to the identity matrix which is unique. The result applies over any matrix (non-singular or singular)

### 4.2.12 THEOREM

Let A be an n x n square non–singular matrix such that

$E_k \cdot E_{k-1} \ \ldots\ldots \ E_2 \cdot E_1 \ A = I$. Then

$E_k \cdot E_{k-1} \ \ldots\ldots \ E_2 \cdot E_1 I_n = A^{-1}$,

where $E_1, E_2, \ldots\ldots, E_k$ are the elementary matrices, pre-multiplied to A, to produce the identity matrix.

**PROOF** Since A is a square non–singular matrix then it can be reduced to the identity matrix $I_n$.

Then

$$E_k \cdot E_{k-1} \ \ldots\ldots \ E_2 \cdot E_1 \ A = I_n$$

$$\Rightarrow \quad E_k \cdot E_{k-1} \ \ldots\ldots \ E_2 \cdot E_1 \ A \cdot A^{-1} = I_n \cdot A^{-1}.$$

$$\Rightarrow \quad E_k \ E_{k-1} \ \ldots\ldots \ E_2 \cdot E_1 \ I_n = A^{-1}.$$

Thus if,

$$[A \mid I_n]$$

is $n \times 2n$ matrix then the elementary operations which produce In place of A produce $A^{-1}$ in the place of $I_n$.

Note that $A = E_1^{-1} E_2^{-1} \ldots\ldots E_{n-1}^{-1} E_n^{-1}$

**EXAMPLE 1**

Find the inverse of $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Take $B = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$

Then $R \sim \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right]$,

by $R_2 + 2R_1$, $R_3 + 4R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right]$$

by $\frac{1}{5} R_2$.

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} -1 & 0 & -3/5 & 1/5 & -2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right]$$

by $R_3 - 6R_2,\ R_1 - 2R_2$.

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} -1 & 0 & -3/5 & 1/5 & -2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & +8 & -6 & +5 \end{array}\right],$$

by $5R_3,\ R_2 + 6/5\ R_3,\ R_1 + 3/5\ R_3,\ -R_1$.

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

**EXAMPLE 2**

Express $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\ (R_3 - R_1)\,A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\ (R_1 + R_3)\,(R_3 - R_1)\,A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\ \frac{1}{2}\,R_3(R_1 + R_3)\,(R_3 - R_1)\,A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\underset{\sim}{R}\ (R_3 + R_2)\left(\frac{1}{2}\,R_3\right)(R_1 + R_3)\,(R_3 - R_1)\,A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\ (R_1 - R_3)\,(R_3 + R_2)\,\frac{1}{2}\,R_3(R_1 + R_3)\,(R_3 - R_1)\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\ (R_2 - 2R_3)\,(R_1 - R_3)\,(R_3 + R_2)\,\frac{1}{2}\,R_3(R_1 + R_3)\,(R_3 - R_1)\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\ E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$\Rightarrow A^{-1} = E_6 E_5 E_4 E_3 E_2 E_1 \quad \text{and}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

## 4.2.13 RANK OF A MATRIX

The concept of rank of a matrix is a useful tool for further use of matrices in different topics of study of linear algebra.

Let $A = [a_{ij}]$ be an m x n matrix over the real field R. Each of the m rows of A is a row vector of the vector space $R^m$ and each of the n columns of A is a column vector of the vector space $R^n$.

## 4.2.14 DEFINITION

The row-rank of a matrix $[a_{ij}]_{mxn}$ is dimension of the row-space generated by the row vectors of the vector space $R^m$, where row-space is a subspace of $R^m$ generated by rows of A, similarly column-rank of the matrix A is dimension of the column space generated by n column vectors of A.

The column-space of A is a subspace of $R^n$, generated by column vectors of A.

## 4.2.15 REMARKS

1. The row-rank of a matrix A is invariant under the elementary row operation and the column-rank is unchanged under the elementary column operation on A.
2. Row-rank of A is the number of linearly independent rows of A and column-rank of A is equal to the number of linearly independent columns of A.

3. As A and $A^t$ have the same rank, therefore row-rank of A is same as that of column-rank of A, which is called the **rank of the matrix A.**

4. As the number of non-zero rows of a matrix A written in its echelon form are linearly independent, therefore, **the number of non-zero rows in echelon form of a matrix A determine the rank of A.**

### 4.2.16 THEOREM

Let A be an n x n matrix. Then A is non–singular if and only if rank of A is n.

**PROOF** Since A can be reduced to the Identity matrix $I_n$, in which each of the rows of $I_n$ is linearly independent, they then form a space of dimension n. Hence the theorem follows.

**EXAMPLE** Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 3 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ -1 & -5 & -3 & -7 & 3 \end{bmatrix}$$

By the application of elementary row operation on A, we get that,

$$R(A) = B = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & -1 & 3 \\ 0 & 3 & 1 & 1 & 1 \\ -1 & -5 & -3 & -7 & 3 \end{bmatrix}, \text{ by interchanging } R_1 \text{ and } R_2$$

$$\text{Then, } R(B) = C = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -3 & -2 & -5 & 3 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -2 & -5 & 3 \end{bmatrix}, \text{ by } R_2-2R_1, R_4+R_1$$

$$R(C) = C^* = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -2 & -5 & 3 \\ 0 & -3 & -2 & -5 & 3 \end{bmatrix}$$ by interchanging $R_2$ and $R_3$.

$$R(C^*) = B^* = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 & 4 \\ 0 & 0 & -1 & -4 & 4 \end{bmatrix}$$ by $R_3+R_2$, $R_4+R_2$.

$$R(B^*) = A^* = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ by $R_4 - R_3$.

$A^*$ contains three non-zero rows and $A^*$ is in echelon form, therefore rank of $A^*$ = rank $A = 3$.

**NOTE**

About a $n \times n$ matrix A, it all sums up to the following equivalent statements;

(i) A is non-singular.

(ii) Rank of A is n.

(iii) Reduced echelon form of A is the identity matrix $I_n$.

(iv) A is a product of elementary matrices.

## EXERCISE 4.2

1. Which of the following matrices is in echelon form? If a matrix is in echelon form, identify the pivots:

(i) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 5 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Which of the matrices in Problem 1 are in reduced echelon form.

3. Complete each of the following statements;

(i) A square matrix is upper triangular if .....

(ii) A square matrix is singular if .....

(iii) A square matrix is non–singular if, .....

(iv) The three elementary row operations are .....

(v) A matrix is in echelon form if .....

(vi) A reduced echelon matrix is .....

(vii) A pivot in an echelon matrix is .....

**4.** Complete each of the following statements;

(i) The column-space of a matrix A is .....

(ii) The row-space of a matrix A is .....

(iii) The dimension of row-space of a matrix A is .....

(iv) The dimension of column-space of a matrix A is .....

**5.** (i) Reduce the following matrices into their echelon forms and hence determine their ranks.

(a) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 1 & 2 \\ 3 & 1 & 2 & -1 \\ 5 & 1 & 0 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 2 & 1 \\ -2 & 2 & 5 & 4 \\ 4 & 0 & -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 & 0 & 2 & 3 \\ 1 & 1 & -1 & -1 & 2 \\ 3 & 0 & -1 & 1 & -2 \\ 2 & -1 & 0 & 2 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & -1 & 3 & 2 & 1 \\ -1 & 1 & 0 & 1 & 3 & -2 \\ 3 & 1 & -2 & 5 & 7 & 0 \\ -5 & 1 & 2 & -3 & -1 & -4 \\ 1 & 3 & -2 & 7 & 13 & -4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

(ii) Transform the following matrices to their reduced echelon forms;

(a) $\begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 3 & 1 & 3 & 4 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 5 & -4 & 6 \\ 1 & 2 & -2 & 3 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

6. Give the inverse, if it exists, of each of the following matrices and verify.

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ 0 & 3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ -1 & -2 & 0 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 2 & 0 & 3 & 1 \\ 1 & 0 & 3 & -2 & 2 \\ 2 & 0 & 0 & -4 & -2 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & -1 & 2 & 1 & 1 \end{bmatrix}$

**7.** Find the inverses of the following matrices over the finite field of residue classes modulo 5.

(i) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 3 \\ 2 & 2 & 1 \end{bmatrix}$

**8.** Find the inverses of the following matrices as product of the elementary matrices.

(a) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

**9.** Find the inverses of the following over the complex field.

(a) $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & i & -i \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$

**10.** Find X such that,

(i) $\begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix} X = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 6 \\ 3 & 4 & 7 & 0 \end{bmatrix}$

(iii) $N X M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, for $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

**11.** Find the values of $a$ and b if

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2b \\ 5 & -3 & 1 \end{bmatrix}$$

**12.** If $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$ then find $(AB)^{-1}$ and AB.

## 4.3 MATRIX OF A LINEAR TRANSFORMATION

Linear transformations from one vector space into another vector space have already been studied in chapter [3] . In this section, a matrix is associated with a given linear transformation which, by the action of matrix multiplication, behaves uniquely in the same way as the linear transformation between two vector spaces. First, we establish the fact about the uniqueness of a linear transformation with respect to the given basis of the given vector space, in the following theorem;

### 4.3.1 THEOREM

Let T be a linear transformation from a vector space U of dimension n, over the field F, into another vector space V over the same field F. If set $\{v_1, v_2, \ldots, v_n\}$ consists of n vectors of V, then T is a unique transformation from U into V, de ined by

$$T(u_i) = v_i, \qquad u_i \in U \qquad i = 1, 2, 3, \cdots, n.$$

where $u_i$ form a basis of U.

**PROOF** As U is n-dimensional vector space, suppose that set $\{u_1, u_2, \cdots, u_n\}$ forms a basis of U. Then every vector $u \in U$, has a unique expression,

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n, \ \alpha_i \in F, \ i = 1, 2, \cdots, n.$$

$$\text{Then } T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n)$$

$$= T(\alpha_1 u_1) + T(\alpha_2 u_2) + \ldots. + T(\alpha_n u_n), \text{ as T is linear.}$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots\cdots + \alpha_n T(u_n)), \text{ as T is linear.}$$

$$= a_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n.$$

If $g : U \longrightarrow V$, is another linear transformation, defined by

$$g(u_i) = v_i, \quad i = 1, 2, \ldots, n.$$

$$\text{Then } g(u) = g(\alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n)$$

$$= \alpha_1 g(u_1) + \alpha_2 g(u_2) + \cdots\cdots + \alpha_n g(u_n)$$

$$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n$$

$$= T(u)$$

This holds for all $u \in U$. Hence $g = T$ by definition, which proves that T is a unique such linear transformation.

### 4.3.2 THEOREM

Let $\{u_1, u_2, \cdots\cdots, u_n\}$ be a basis of a vector space W over the field F and V be another vector space over the same field F. If $\{v_1, v_2, \cdots\cdots, v_n\}$ be a set of n vectors of V and $T : W \longrightarrow V$, be a linear transformation from W into V defined by $T(u_i) = v_i$, $i = 1, 2, \cdots\cdots n$. Then T is an isomorphism if and only if $\{v_1, v_2, \cdots\cdots, v_n\}$ is a basis of V.

**PROOF** Suppose $\{v_1, v_2, \cdots, v_n\}$ is a basis of V. Then $v \in V$ can be expressed as,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n \quad , \quad \alpha_i \in F, i = 1, 2, \cdots n.$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots\cdots + \alpha_n T(u_n)$$

$$= T(\alpha_1 u_1) + T(\alpha_2 u_2) + \cdots\cdots + T(\alpha_n u_n)$$

$$= T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n), \text{ as T is a linear map}$$

$$= T(u)$$

Which implies, that, for every $v \in V$, there exists $u \in W$, such that $v = T(u)$. Thus, T is a map onto V. Then, for some $u \in W$, $T(u) = 0 \in V$.

$$\Rightarrow \quad T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n) = 0 \in V.$$

$$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n = 0 \in V.$$

Since $\{v_1, v_2, \cdots, v_n\}$ is a basis of V, therefore,

$$\alpha_1 = \alpha_2 = \cdots\cdots = \alpha_n = 0$$

or $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n = 0 \in W.$

Hence Ker T = {0} and T is (1–1). Therefore T is an isomorphism.

Conversely, if T is (1 – 1), then $T(0_W) = 0_V$ and hence Ket T = 0. Also

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_n u_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots\cdots = \alpha_n = 0$$

Where $\alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots\cdots = \alpha_n = 0$$

Which proves that $\{v_1, v_2, \ldots, v_n\}$ is a basis of V because T is onto V and every element of V is a linear combination of $\{v_1, v_2, \ldots, v_n\}$.

Now we associate a matrix, which is unique, with a linear transformation T from a vector space U of dimension m into another vector space V of dimension n, over the same field F.

Suppose $\{u_1, u_2, \cdots\cdots, u_m\}$ and $\{v_1, v_2, \cdots\cdots, v_n\}$ form a basis of U and V respectively. Any vector $u \in U$ is uniquely expressed as a linear combination of the vectors $\{u_1, u_2, \cdots\cdots u_m\}$

$$\text{i.e. } u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots\cdots + \alpha_m u_m, \quad \alpha_i \in F, \quad i = 1, 2, \cdots, m.$$

$(\alpha_1, \alpha_2, \cdots, \alpha_m)$ are called the coordinates of u relative to the basis $\{u_1, u_2, \cdots, u_m\}$ of U.

A mapping $T : U \longrightarrow V$, has its images.

$T(u_1), T(u_2), \cdots\cdots, T(u_m)$ in V, of the basis vectors $u_1, u_2, \cdots\cdots, u_m$ of U.

As $\{v_1, v_2, \cdots, v_n\}$ are basis vectors of V, therefore, vectors $T(u_1), T(u_2), \ldots, T(u_m)$ have their linear combination of the vectors $\{v_1, v_2, \cdots\cdots, v_n\}$ as,

$$\left.\begin{aligned} T(u_1) &= a_{11} v_1 + a_{12} v_2 + \cdots\cdots + a_{1n} v_n \\ T(u_2) &= a_{21} v_1 + a_{22} v_2 + \cdots\cdots + a_{2n} v_n \\ T(u_m) &= a_{11} v_1 + a_{m2} v_2 + \cdots\cdots + a_{mn} v_n \end{aligned}\right\} \quad \cdots\cdots \quad (a)$$

where $a_{ij} \in F, \quad 1 \le i \le m, \ 1 \le j \le n.$

The matrix associated with T is that, whose columns are formed by the coordinates of the images $T(u_1), T(u_2), \cdots, T(u_m)$ in the image space V w.r.t its basis $\{v_1, v_2, \cdots, v_n\}$

Thus $A = \begin{bmatrix} a_{11} & a_{21} & \vdots & a_{m1} \\ a_{12} & a_{22} & \vdots & a_{m2} \\ :: & :: & \vdots & :: \\ a_{1n} & a_{2n} & \vdots & a_{mn} \end{bmatrix}$

is the matrix called, the matrix of linear transformation T.

Note that (i) The matrix A is a $n \times m$ matrix.

(ii) A is infact transpose of the coefficient matrix of the system of equations given in (a).

(iii) If U and V are of equal dimensions n then A is a $n \times n$ square matrix.

**EXAMPLE** A linear transformation $T : R^2 \to R^3$ is defined by

$$T(x_1, x_2) = (2x_1 - 3x_2, 3x_1 - x_2, x_1 + x_2).$$

Then the matrix of the transformation w.r.t the standard basis of $R^2$ and $R^3$ is

$$A = \begin{bmatrix} 2 & -3 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}, \text{ because}$$

$$T(e_1) = T(1, 0) = (2, 3, 1).$$

$$= 2(1, 0, 0) + 3(0, 1, 0) + 1(0, 0, 1)$$

$$= 2e_1 + 3e_2 + 1 \cdot e_3$$

$T(e_1)$ has coordinates (2, 3, 1), which form the first column of A.

$$T(e_2) = T(0, 1) = (-3, -1, 1)$$

$$= -3(e_1) - 1(e_2) + 1e_3$$

Thus, Coordinates of $T(e_2)$ are (–3, –1, 1), which form the second column of A.

Note that, $T(e_1) = A(e_1) = \begin{bmatrix} 2 & -3 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, by actual pre-multiplication

with A to the basis vectors of $R^2$. That is, $T[e_1] = A[e_1], T[e_2] = A[e_2]$.

### 4.3.3. LINEAR TRANSFORMATION WHEN ITS MATRIX IS GIVEN

Let A be an m x n matrix associated with a linear transformation from $R^n$ into $R^m$. We write

$$A = \begin{bmatrix} (a_1) \\ (a_2) \\ \vdots \\ (a_m) \end{bmatrix}, \text{ where } (a_1), (a_2), \ldots, (a_m) \text{ be the row vectors of A which}$$

belong to $R^n$. The mapping T from $R^n \to R^m$ whose matrix is A, will be defined by the formula

$$T(x) = (a_1 \cdot x, a_2 \cdot x, \cdots, a_m \cdot x)$$

Thus T(x) is the vector whose entries are the dot products of the row vectors of A with (x). Thus map T is associated with the matrix A.

For example,

If $R^2 \to R^3$ is a linear map associated with the matrix $A = \begin{bmatrix} -1 & 4 \\ 5 & -3 \\ 2 & 0 \end{bmatrix}$,

relative to the standard basis of $R^2$ and $R^3$ and

$x = (x_1, x_2) \in R^2$. Then

$$T(x) = ((-1, 4)\cdot(x_1, x_2), (5, -3)\cdot(x_1, x_2), (2, 0)\cdot(x_1, x_2))$$

$$= (-x_1 + 4x_2, 5x_1 - 3x_2, 2x_1)$$

defines the linear transformation associated with A.

### 4.3.4 DEFINITION

The matrix of a linear transformation $T : R^n \longrightarrow R^m$, w.r.t a basis $\{u_1, u_2, \cdots, u_n\}$ of $R^n$ and $\{v_1, v_2, \cdots, v_m\}$ of $R^m$ is the m x n matrix A whose ith column is formed by the coordinates of $T(v_i)$.

$$A = [CT(v_1) \quad CT(v_2), \cdots\cdots, CT(v_n)]$$

**NOTE** It may be noted that for a linear transformation $T : U \longrightarrow V$, relative to a different choice of bases of U and V, entirely a different matrix is associated with the transformation T.

### 4.3.5 EXAMPLES

**1.** Find the matrix of the linear transformation $T : R^3 \longrightarrow R^3$, defined by,

$$T(x, y, z) = (y, -x, -z)$$

relative to the standard basis

$$e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)$$

Take $T(e_1) = T(1, 0, 0) = (0, -1, 0) = 0e_1 - 1e_2 + 0e_3$

$T(e_2) = T(0, 1, 0) = (1, 0, 0) = 1e_1 + 0e_2 + 0e_3$

$T(e_3) = T(0, 0, 1) = (0, 0, -1) = 0(e_1) + 0e_2 - 1e_3.$

As the components of $T(e_1), T(e_2), T(e_3)$ form the columns of A, therefore,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ is the required matrix.

**2.** Find the matrix associated with the mapping $T : R^3 \longrightarrow R^2$, defined by

$$T(x, y, z) = (3x - 4y + 9z, 5x + 3y - 2z)$$

relative to the standard basis of $R^3$ and $R^2$.

Take $T(e_1) = T(1, 0, 0) = (3, 5) = 3(1, 0) + 5(0, 1)$

$T(e_2) = T(0, 1, 0) = (-4, 3) = -4(1, 0) + 3(0, 1)$

$T(e_3) = T(0, 0, 1) = (9, -2) = 9(1, 0) - 2(0, 1)$

Thus the matrix A of T is $\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$

**3.** Let $T : R^3 \longrightarrow R^2$ be a linear transformation defined by $T(x_1, x_2, x_3) = (x_1, -3x_1 + x_3)$. Then find the matrix of T relative to the basis $\{(1, 0, 3), (1, 2, 5), (1, 3, 2)\}$ of $R^3$ and the standard basis of $R^2$.

Take $T(1, 0, 3) = (1, -3 + 3) = (1, 0) = 1(1, 0) + 0(0, 1)$

$T(1, 2, 5) = (1, -3 + 5) = (1, 2) = 1(1, 0) + 2(0, 1)$

$T(1, 3, 2) = (1, -3 + 2) = (1, -1) = 1(1, 0) - 1(0, 1)$

Thus the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ is the matrix of T defined in the exercise.

4. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 5 & 3 \\ 2 & 6 & -1 \end{bmatrix}$ is the matrix of $T : R^3 \longrightarrow R^3$ relative to the standard basis. Determine T in terms of the coordinates of $u \in R^3$.

Take $u = (u_1, u_2, u_3) \in R^3$. If T is the linear transformation of A, then

$$T(u_1, u_2, u_3) = ((u_1, u_2, u_3)\cdot(1,2,-2),(u_1, u_2, u_3)\cdot(1,5,3),(u_1, u_2, u_3)\cdot(2,6,-1))$$

$$= (u_1 + 2u_2 - 2u_3, u_1 + 5u_2 + 3u_3, 2u_1 + 6u_2 - u_3).$$

$$A\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 5 & 3 \\ 2 & 6 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (u_1 + 2u_2 - 2u_3, u_1 + 5u_2 + 3u_3, 2u_1 + 6u_2 - u_3)$$

$$\Rightarrow T(u_1, u_2, u_3) = (u_1 + 2u_2 - 2u_3, u_1 + 5u_2 + 3u_3, 2u_1 + 6u_2 - u_3)$$

5. Let $T : R^3 \longrightarrow R^3$ be a linear transformation whose matrix is

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$ relative to the standard basis of $R^3$.

Find the matrix of T relative to the basis $\{(0,1,2), (1,1,1), (1,0,2)\} = B$.

Take $u = (u_1, u_2, u_3) \in R^3$. Then the transformation of the matrix A is defined by,

$$T(u_1, u_2, u_3) = (u_2 - u_3, u_1 - u_3, u_1 - u_3)$$

Now we find the matrix of T relative the basis B. Then

$T(0, 1, 2) = (1 - 2, 0 - 2, 0 - 2) = (-1, -2, -2).$

$(-1, -2, -2) = \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$

$(-1, -2, -2) = (\beta + \gamma, \alpha + \beta, 2\alpha + \beta - 2\gamma)$ are the coordinates of $T(0, 1, 2)$ relative to the basis B.

To find $\alpha, \beta, \gamma$, we solve simultaneously, the equations

$$\beta + \gamma = -1 \; ; \quad \alpha + \beta = -2; \quad 2\alpha + \beta - 2\gamma = -2$$

$$(\alpha + \beta) - (\beta + \gamma) = -2 + 1$$

$$\alpha + \beta - \beta - \gamma = -1$$

$$\Rightarrow \alpha - \gamma = -1$$

put in $2\alpha + \beta - 2\gamma = -2$ and get,

$$2(\alpha - \gamma) + \beta = -2$$

$$2(-1) + \beta = -2 \Rightarrow \beta = -2 + 2 = 0$$

Then $\alpha = -2 - \beta = -2$ and $\gamma = -2 + 1 = -1$

and $(\alpha, \beta, \gamma) = (-2, 0, -1)$

For the coordinates of T(1, 1, 1), take

$$T(1, 1, 1) = (0, 0, 0)$$

$$(0, 0, 0) = \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$$

$$(0, 0, 0) = (\beta + \gamma, \alpha + \beta, 2\alpha + \beta - 2\gamma)$$

$$\left.\begin{array}{l} \beta + \gamma = 0 \Rightarrow \beta = -\gamma, \\ \beta + \alpha = 0 \Rightarrow \beta = -\alpha \end{array}\right] \Rightarrow \alpha = \gamma$$

From $2\alpha + \beta - 2\gamma = 0 \Rightarrow \beta = 0 = \alpha = \gamma$.

Coordinates of T(1, 1, 1) are (0, 0, 0).

Similarly, $T(1, 0, -2 = (2, 3, 3)$

$$(2, 3, 3) = \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$$

$$\left.\begin{array}{l} \Rightarrow \beta + \gamma = 2 \\ \alpha + \beta = 3 \end{array}\right] \Rightarrow \alpha - \gamma = 1$$

$$2\alpha + -2\gamma = 3 \Rightarrow 2.1 + \beta = 3$$

$$\Rightarrow \quad \beta = 1$$

Hence $\gamma = 1$, $\alpha = 2$.

The coordinates of $T(1, 0, -2)$ are $(\alpha, \beta, \gamma) = (2, 1, 1)$ so the matrix of the transformation is

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

6. Let $T : R^3 \longrightarrow R^2$ be a linear transformation, defined by,

$T(x_1, x_2, x_3) = (3x_1 + 2x_2, x_2 - 2x_3)$

Find the matrix of T with respect to the basis $\{(1, 0, 1), (1, 2, 1), (0, 1, 2)\}$ of $R^3$ and $\{(1, 2), (1, 1)\}$ of $R^2$.

Take $T(1, 0, 1) = (3, -2) = -5(1, 2) + 8(1, 1)$

$T(1, 2, 1) = (7, 0) = -7(1, 2) + 14(1, 1)$

$T(0, 1, 2) = (2, -3) = -5(1, 2) + 7(1, 1)$

The coordinates of $T(1, 0, 1)$, $T(1, 2, 1)$, $T(0, 1, 2)$ are

$$(-5, 8), (-7, 14) \text{ and } (-5, 7)$$

Therefore, the matrix of the linear transformation T is

$$\begin{bmatrix} -5 & -7 & -5 \\ 8 & 14 & 7 \end{bmatrix}$$

7. Find a formula and the matrix of the linear transformation

$T : R^3 \longrightarrow R^3$, defined by,

$T(1, 0, 0) = (1, 1, 0)$

$T(0, 1, 0) = (2, 0, 1)$ and $T(0, 0, 1) = (1, 1, 1)$.

relative to the standard basis.

Clearly, $T(1, 0, 0) = (1, 1, 0) = 1(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1)$

$$T(0, 1, 0) = (2, 0, 1) = 2(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

$$T(0, 0, 1) = (1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$$

Thus the matrix of T is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and the formula is $(u_1 + 2u_2 + u_3, u_1 + u_3, u_2 + u_3) = T(u_1, u_2, u_3)$

for $u = (u_1, u_2, u_3) \in R^3$.

**8.** Let $T : R^2 \longrightarrow R^3$ be a linear transformation with the bases

$\{(1, 0), (1, 1)\}$ of $R^2$ and $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ of $R^3$.

Then find the matrix of T. Find also the formula of transformation

$$T(u_1, u_2), \quad u = (u_1, u_2) \in R^2.$$

If $T(1, 0) = (1, 0, 1)$, $T(1, 1) = (1, 1, 2)$

To find the matrix of T,

Find the coordinates of T(1,0) and T(1,1) with respect to the given basis of $R^3$

i.e. $T(1, 0) = (1, 0, 1) = 1(1, 0, 0) - 1(1, 1, 0) + 1(1, 1, 1).$

$T(1, 1) = (1, 1, 2) = 0(1, 0, 0) - 1(1, 1, 0) + 2(1, 1, 1).$

Thus the matrix of T is $\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \end{bmatrix}$.

To find the formula of T, take $(u_1, u_2) \in R^2$. Then change $(u_1, u_2)$ into the given basis $\{(1, 0), (1, 1)\}$

It becomes $(u_1 - u_2, u_2)$ as coordinates of $(u_1, u_2)$ under the new basis.

i.e. $(u_1, u_2) = (u_1 - u_2)(1, 0) + u_2(1, 1)$

Now $T(u_1, u_2) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 - u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ -u_1 \\ u_1 + u_2 \end{bmatrix}$

$(u_1 - u_2, -u_1, u_1 + u_2)$ are the coordinates of the vector $T(u_1, u_2)$. Thus

$$\begin{aligned} T(u_1, u_2) &= (u_1 - u_2)(1, 0, 0) + u_1(1, 1, 0) + (u_1 + u_2)(1, 1, 1) \\ &= (u_1 - u_2 - u_1 + u_1 + u_2, -u_1 + u_1 + u_2, u_1 + u_2) \\ &= (u_1, u_2, u_1 + u_2) \end{aligned}$$

is the rule of transformation T.

Thus it completes the answer.

## EXERCISE 4.3

1. Find the coordinates of (2, 1, 3) with respect to;

   (i) The basis, {(1, 0, 1), (1, 1, 0), (1, 1, 1)} of $R^3$.

   (ii) The basis, {(0, 0, 1), (0, 1, 1), (1, 1, 1)} of $R^3$.

2. Find the standard matrix for the linear transformation $T : R^3 \longrightarrow R^3$ defined in each part.

   (i) $T(1, 0, 0) = (1, 1, 0)$; $T(0, 1, 0) = (2, 0, 1)$; $T(0, 0, 1) = (1,1,1)$

   (ii) $T(1, 2, 1) = (-1, 2, -3)$; $T(2,1,1) = (2,-1,1)$; $T(0, 0,1) = (0, 0, 3)$

   (iii) $T(1, 2, 3) = (1, 1, -5)$; $T(-1,0,-1) = (5,-1,1)$; $T(1,3,2) = (0,1,0)$

3. Let $T : R^2 \longrightarrow R^3$ be a linear transformation relative to the basis

   {(1, 0), (1, 1)} of $R^2$ and {(1, 0, 0), (1, 1, 0), (1, 1, 1)} of $R^3$ defined by

   $$\begin{aligned} T(1, 0) &= (1, 0, 1) \\ T(1, 1) &= (1, 1, 2). \end{aligned}$$

   Find the matrix of T.

4. Let $T : R^3 \to R^2$ be a defined by

   $$T(x_1, x_2, x_3) = (3x_1 + 2x_2, x_2 - 2x_3)$$

Find the matrix of T with respect to the bases

$\{(1, 0, 1), (1, 2, 1), (0, 1, 2)\}$ of $R^3$ and $\{(1, 2), (4, 0)$ of $R^2$.

**5.** Let T be linear transformation defined as in Q4. Find the matrix for T with respect to the bases,

$\{(0, 0, 1), (0, 1, 1), (1, 1, 1)$ for $R^3$ and $\{(1, 0), (1, 1)\}$ for $R^2$.

**6.** Let $T : R^2 \to R^3$ be defined by $T(x,y) = (x-y, 2x+y, -x+3y)$. Find the matrix of T w.r.t the bases $\{(3, 2), (4, 1)\}$ of $R^2$ and $\{(1,1,1), (0,1,1), (0,0,1)\}$ for $R^3$

**7.** Show that the matrix of $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (3x_1 + 2x_2 + x_3, x_2 - x_3)$ with respect to the standard basis for $R^3$ and bases $\{(2, 1), (1, -1)\}$ of $R^2$ is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

**8.** A linear mapping $T : R^2 \to R^3$ be defined by

$$T(1, 1) = (0, 1, 2); \qquad T(-1, 1) = (2, 1, 0).$$

Find the matrix of T relative to the standard bases. Find also the rule of definition for T.

**9.** Let $T : R^4 \to R^3$ be defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 + x_3 - x_4, x_3 - x_4)$$

Find the matrix of T with respect to the basis

$\{(1,1,1,1), (1,1,1,-1), (1,1,-1,-1), (1,-1,-1,-1)\}$ of $R^4$

and the basis $\{(2,0,0), (0,1,0), (0,0,-2)\}$ of $R^3$.

**10.** Find a formula and the matrix for the linear transformation $T : R^3 \to R^2$, defined by

$T(1,0,3) = (1,0)$ ; $T(1,2,5) = (1,2)$ ; $T(1,3,2) = (1,-1)$

with respect to the basis $\{(1,0,3), (1,2,5), (1,3,2)\}$ of $R^3$ and the standard basis of $R^2$.

**11.** Let $T : V \longrightarrow V$ be a linear transformation defined by $T(p(x)) = \frac{d}{dx}(p(x))$,

where V is vector space of all polynomials p(x) in x with real coefficients of degree not exceeding 3. Find the matrix of T relative to the bases,

$\{1, x, x^2, x^3\}$ and $\{1, 1 + x, 1 + x^2, 1 + x^3\}$ of V.

12. If V is the vector space of all polynomials of degree not exceeding 2 and T is defined as in Q11, then find the matrix of T w.r.t the basis $\{1, 1+x, 1+x^2\}$.

## 4.4 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

Let $x_1, x_2, \ldots\ldots, x_n$ be n variable quantities in relation linear to each other, with the following system of m distinct linear equations;

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n &= b_m \end{aligned} \qquad \ldots\ldots \text{(i)}$$

with coefficient $a_{ij}$, $(1 \le i \le m, 1 \le j \le n)$ as real numbers.

Their simultaneous solution, we mean by, the real replacements of x's which satisfy each equation of the system (i). The students are already familiar with the elimination method of solving such a system of linear equations simultaneously. We use here the matrix method after expressing the system (i) into its matrix form as;

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \ldots\ldots \text{(ii)}$$

or $\quad Ax = b \qquad \ldots\ldots$ (iii)

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$ is the coefficient m x n matrix.

$x = [x] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is n x 1 and $b = [b] = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, an m x 1 matrix.

Note that the system (i) of linear equations is expressed in its matrix form (ii) under the matrix multiplication of a m x n matrix A with an n x 1 matrix [x] to produce m x 1 matrix [b]. The system (i) is called homogeneous if its each equation is homogeneous.

i.e. $b_i = 0$ , for each i

and will be written in its matrix form as, ..... (iv)

$$Ax = \mathbf{0}$$

where $\mathbf{0}$ is m x 1 zero matrix.

If x and b in the system (ii) are column vectors of the vector spaces $R^n$ and $R^m$ respectively, then A acts as a linear transformation from vector space $R^n$ into $R^m$. The system is completely determined by the coefficient matrix A and the vector b.

### 4.4.1 DEFINITION

The system (i) of m linear equations involving n unknowns is said to be **consistent** if it has a solution, otherwise it is called **inconsistent.**

In general, the system (i) may or may not have a solution. If $m \neq n$, then either $Ax = b$ has no solution or has infinite number of solutions. If $m < n$ then the system has infinite number of solutions.

If $m = n$, then the system (ii) has a unique solution, if in particular, A is a non-singular matrix. In this case all the rows or columns of A form L.I. set of vectors of $R^n$.

### 4.4.2 THEOREM

The system $Ax = b$, with $m = n$ has a unique solution if A is non-singular.

**PROOF** Suppose that A is non-singular. Then the inverse $A^{-1}$ of A exists which is unique. Thus,

$$Ax = b$$
$$\Rightarrow A^{-1}(Ax) = A^{-1}b$$
$$\Rightarrow (A^{-1}A)x = A^{-1}b \quad \text{(Associative property).}$$
$$\Rightarrow I_n x = A^{-1}b \quad \text{(Inverse property).}$$
$$\Rightarrow x = A^{-1}b \quad \text{(Identity property).}$$

Hence vector x is a unique solution of the system $Ax = b$.

If A is singular then $A^{-1}$ does not exist and consequently x does not exist as a unique solution.

**COR.** If the system $Ax = b$ is homogeneous then $b = \mathbf{0}$

and $Ax = \mathbf{0} \Rightarrow A^{-1}(Ax) = A^{-1}\mathbf{0}$

$$\Rightarrow (A^{-1}A)x = 0$$
$$\Rightarrow I_n x = 0$$
$$\Rightarrow x = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is a trivial solution of the system.

The condition for existence of a nontrivial solution of a homogeneous system $Ax = 0$ follows by the following theorem;

### 4.4.3 THEOREM

A system $Ax = 0$ of n homogeneous linear equations in n variables has a non-trivial solution if and only if rank(A) is less than n.

PROOF Suppose that $k < n$ be the rank of A. Then the column-space of A is of dimension k. Thus any subset of columns of A containing $(k + 1)$ columns is linearly dependent. If all of those linearly independent k columns be placed in the first k places of the columns of A then the remaining $(n - k)$ columns are linear combinations of the first k columns of A. In the $(n - k)$ places, arbitrary columns can be placed and non-trivial solution of the system can thus be obtained.

Conversely, suppose that $Ax = 0$ has a non-trivial solution $x_0$ say, and rank of A is n, then $A^{-1}$ exists

and $Ax_0 = 0 \Rightarrow A^{-1}(Ax_0) = 0$
$$\Rightarrow (A^{-1}A)x_0 = 0$$
$$\Rightarrow (I_n)x_0 = 0$$
$$\Rightarrow x_0 = 0$$

which provides contradiction to the assumption that $x_0$ is nontrivial. Hence rank of A is less than n.

### 4.4.4 CONSISTENCY CRITERION FOR A SYSTEM

In a system of m linear equations involving n variables, $Ax = b$, A is a $m \times n$ matrix, x is $m \times 1$ and b an $m \times 1$ matrix over the real field R.

Then the matrix $[A|b]$, obtained by adjoining the column vector b to the matrix A on the right, is called the **augmented matrix** of the system $Ax = b$.

### 4.4.5 THEOREM

The system $Ax = b$ of m linear equations involving n variables is consistent if and only if the rank of the augmented matrix $[A|b]$ is equal to the rank of A.

**PROOF** Let A be an m x n matrix and rank of A is r. Then A has r linearly independent column vectors. Without loss of generality, we may take first r columns $C_1, C_2, C_3 \ldots\ldots C_r$ to be linearly independent. Then for each $j = r+1, r+2, \ldots, n$, $C_j$ is a linear combination of $C_1, C_2, \ldots C_r$. Now [A | b] has just one column, namely (b), in addition to the n columns of A. Hence the maximal number of linearly independent column vectors in [A | b] is either r or (r+1). Therefore rank [A | b] = r or r + 1.

Suppose rank [A | b] = r, then $C_1, C_2, \ldots, C_r$ are linearly independent column vectors of [A | b]. Therefore, there exist scalars $\alpha_1, \alpha_2, \ldots\ldots \alpha_{r+1}$ (not all zero) such that

$$\alpha_1 C_1 + \alpha_2 C_2 + \ldots\ldots + \alpha_{r+1} \cdot b = 0$$

Now $\alpha_{r+1} \neq 0$, otherwise $C_1, C_2, \ldots\ldots, C_r$ would be linearly dependent, contrary to our assumption. Hence

$$b = -\frac{\alpha_1}{\alpha_{r+1}} C_1 - \ldots\ldots\ldots\ldots - \frac{\alpha_r}{\alpha_{r+1}} C_r$$

$$= A\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ where } \beta_j = \frac{-\alpha_j}{\alpha_{j+1}}, \quad j = 1, 2, \ldots\ldots r.$$

Thus $x = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of the given system and so the system is consistent.

Conversely, suppose that the system $Ax = b$ is consistent and let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ be a solution. Then}$$

$$b = A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 C_1 + x_2 C_2 + \ldots\ldots + x_r C_r.$$

Now $C_j, j = r+1, r+2, \ldots, n$ is linear combination of $C_1, C_2, \ldots, C_r$. Hence b is also a linear combination of $C_1, C_2, \ldots, C_r$. Hence the set $\{C_1, C_2, \ldots, C_r, b\}$ is linearly dependent. Consequently [A | b] has only r linearly independent column vectors and rank [A | b] = r.

COR. If A is an m x n matrix of rank m, then the system $Ax = b$ is consistent.

### 4.4.6 THEOREM

Let the system $Ax = b$ be consistent. The general solution to the non-homogeneous system $Ax = b$ is a particular solution of the non-homogeneous system plus the general solution to the associated homogeneous system $Ax = \mathbf{0}$.

PROOF Let $x_p$ be a particular solution of the system $Ax = b$ and $x_0$ be a solution of the corresponding homogeneous system $Ax = \mathbf{0}$. Then

$$Ax_p = b \text{ and } Ax_0 = \mathbf{0}$$

$$A(x_p + x_0) = Ax_p + Ax_0 = b + 0 = b$$

which gives that $x_p + x_0$ is a solution of $Ax = b$.

Let $x'$ be any solution of the system $Ax = b$. Then there is a solution $x_0$ of the homogeneous system $Ax = \mathbf{0}$,

where $x' = x_p + x_0$

and $x_0 = x' - x_p$

when $Ax_0 = A(x' - x_p) = Ax' - Ax_p = \mathbf{0} - \mathbf{0} = \mathbf{0}$.

REMARK This theorem suggests that a consistent system $Ax = b$ has a unique solution if and only if its associated homogeneous system $Ax = 0$ has a unique solution. But $Ax = 0$ has a unique solution if and only if A is a square non-singular matrix.

### 4.4.7 VALUABLE COMMENTS

1. The system $Ax = b$ of m linear equations involving n variables where A is an m x n matrix over the field R. Then

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots\ldots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \ldots\ldots (v)$$

where the vector $b \in R^m$ is a linear combination of the column vectors of A. Hence the above system is consistent, if there exist $x_1, x_2, \cdots, x_n \in R$ which satisfy (iv), that is, b is a linear combination of the column vectors of A.

2. Assuming that the system $Ax = b$ is consistent and rank of A is r, then general solution assigns arbitrary values to $(n - r)$ of the unknowns. Hence the solution is unique if and only if $n - r = 0$, that is $n = r$. It means that n columns of A are linearly independent.

3. Combining (1) and (2) together one concludes that the system $Ax = b$ has a unique solution if and only if the columns of A are linearly independent and b is a linear combination of them.

4. In particular, if A is a square matrix (m = n), then columns of A are linearly independent if and only if A is non-singular. In this case the n linearly independent column vectors of $R^n$ span $R^n$ so that b is automatically a linear combination of them.

5. Each of the three elementary row operations transforms the system $Ax = b$ into an equivalent system $A^*x = b^*$, under which the solution remains invariant.

## 4.4.8 SOLUTION OF THE SYSTEM OF EQUATIONS

We have discussed the theory behind the existence of a solution or solutions or no solution at all of a given system of equations $Ax = b$. In this part, we perform the methods which usually help to solve a given system of equations (homogeneous or non-homogeneous). There are two methods used for this purpose (i) Gauss elimination method and (ii) Gauss-Jordan elimination procedure. Both are associated with the name of a very famous mathematician, for all the times, named K.E. Gauss (1777-1855).

### (i) GAUSS ELIMINATION METHOD

Under this procedure, the given system is expressed in its matrix form $Ax = b$. Then by the successive application of the elementary row operations over the system to produce $A^*x = b^*$, where $A^*$ is in reduced echelon form of the matrix A of the given system. Thus if

$$R(A) = A^* \quad \text{and} \quad R(b) = b^*$$

Then we write the system of equations as,

$$A^*x = b^*$$

which gives x easily. This method is generally used in solving the system of equations given by matrix multiplication technique.

**(ii)** The alternate method is called **Gauss-Jordan method**. In this procedure, the pivot equation is first divided by the coefficient of the pivot variable. Then the pivot variable is eliminated from the prior equations as well as the subsequent ones. By experience one learns that Gauss–Jordan method requires more multiplications but requires much less writing than the procedure involved by Gauss elimination method. Infact, either of the two will complete the solution, in almost a similar fashion. It is therefore suggested that routine calculations could be done by either of the methods. The augmented matrix. [A|b] is the centre of convergence of elimination instead of the system $Ax = b$.

**REMARK**

1. Every matrix of the given system can be reduced to its row equivalent echelon matrix, which is rather uniquely determined.
2. The reason that Gaussian elimination method works is that row–operations are reversible. It follows that solution of the simplified system is same as that of the given system.

**NOTE**

About a $n \times n$ matrix A it sums up to the equality of the following statements;

(i) A is non - singular.

(ii) The homogeneous system $Ax = 0$ has only the trivial (or zero) solution.

(iii) The linear system $Ax = B$ has a unique solution for every $n \times 1$ matrix B.

**EXAMPLE 1** Solve the system of equations;

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 + 2x_2 + 3x_3 + x_4 &= 5 \\ x_1 - x_2 + 2x_3 + 3x_4 + x_5 &= 4 \end{aligned}$$

for real $x_1, x_2, x_3, x_4, x_5$.

**SOLUTION** The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & 2 & 3 & 1 & 0 \\ 1 & -1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Its augmented matrix

$$A_b = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 2 \\ 2 & 2 & 3 & 1 & 0 & 5 \\ 1 & -1 & 2 & 3 & 1 & 4 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & -2 & 1 & 4 & 1 & 2 \end{array}\right], \quad \text{by } R_2 - 2R_1,\ R_3 - R_1.$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right], \text{ by interchanging } R_3 \text{ and } R_2.$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & -2 & -1/2 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right], \text{ by } -\frac{1}{2}R_2.$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 3/2 & 1 & 1/2 & 3 \\ 0 & 1 & -1/2 & -2 & -1/2 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right], \text{ by } R_1 - R_2.$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -7/2 & 1/2 & 3/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right], \quad \text{by } R_1 - \frac{3}{2}R_3 \text{ and } R_2 + \frac{1}{2}R_3$$

The system which corresponds to the row echelon matrix is,

$$x_1 - 7/2x_4 + \frac{1}{2}x_5 = 3/2$$

$$x_2 - \frac{1}{2}x_4 - \frac{1}{2}x_5 = -1/2$$

$$x_3 + 3x_4 = 1$$

The variables $x_4$ and $x_5$ can be assigned any real values, say $x_4 = c_1$ and $x_5 = c_2$.

$$x_1 = \frac{3}{2} + \frac{7}{2}c_1 - \frac{1}{2}c_2$$

$$x_2 = \frac{-1}{2} + \frac{1}{2}c_1 + \frac{1}{2}c_2$$

$$x_3 = 1 - 3c_1.$$

Thus the solutions are

$$\left(\frac{3}{2} + \frac{7}{2}c_1 - \frac{1}{2}c_2, -\frac{1}{2} + \frac{1}{2}c_1 + \frac{1}{2}c_2, 1 - 3c_1, c_1, c_2\right)$$

This way there are infinitely many solutions of the given system depending on $c_1$ and $c_2$.

**EXAMPLE 2** Solve the system for real x's.

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 7 \\ -x_1 + 4x_2 - 3x_3 &= 4. \end{aligned}$$

**SOLUTION** The system expressed as $Ax = b$ will give

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & 4 & -3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \quad b = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}$$

$$\text{Then } A_b = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 7 \\ -1 & 4 & -3 & 4 \end{array}\right]$$

$$R(A_b) = B = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 6 \\ 0 & 3 & -2 & 5 \end{array}\right], \text{ by } R_2 - R_1 \text{ and } R_3 + R_1$$

$$R(B) = C = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right], \text{ by } R_3 - R_2.$$

$$R(C) = C^* = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -2/3 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right], \text{ by } \frac{1}{3} R_2.$$

$$R(C^*) = B^* = \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 3 \\ 0 & 1 & -2/3 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right], \text{ by } R_1 + R_2.$$

$$R(B^*) = A^* = \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \text{ by } R_1 + 3R_3 \text{ and } R_2 + 2R_3$$

which is of reduced echelon form. $A^*$ then corresponds to the system.

$$\begin{aligned} x_1 + \frac{1}{3}x_3 &= 0 \\ x_2 - \frac{2}{3}x_3 &= 0 \\ 0 &= 1, \end{aligned}$$

which, of course has no solution, since $0 \neq 1$.

**EXAMPLE 3** Solve for real $x_1, x_2, x_3$, the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 7x_2 + x_3 &= 13 \\ x_1 + 4x_2 - 6x_3 &= 1 \end{aligned}$$

**PROOF** The system given in its matrix form is expressed as,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 1 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 1 \end{bmatrix}$$

The Augmented matrix

$$A_b = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 7 & 1 & 13 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

$$R(A_b) = B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & -9 & -5 \end{array}\right], \text{ by } R_2 - 2R_1,\ R_3 - R_1$$

$$R(B) = C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 3 & -5 & 1 \\ 0 & 1 & -9/2 & -5/2 \end{array}\right], \text{ by } \frac{1}{2}R_3.$$

$$R(C) = D = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 3 & -5 & 1 \end{array}\right], \text{ by interchanging } R_2 \text{ to } R_3.$$

$$R(D) = C^* = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 0 & 17/2 & 17/2 \end{array}\right], \text{ by } R_3 - 3R_2.$$

$$R(C_*) = B^* = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 0 & 1 & 1 \end{array}\right], \text{ by } \frac{2}{17}R_3.$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right], \text{ by } R_2 + \frac{9}{2}R_3.$$

$$R(B^*) = A^* = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right], \text{ by } R_1 - 2R_2.$$

$$R(A^*) = E^* = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right], \text{ by } R_1 - 3R_3.$$

We now form the system to be

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Thus the solution vector $(x_1, x_2, x_3) = (-1, 2, 1)$ is unique.

**EXAMPLE 4**

Inverstigate, for what values of $a$, b, the simultaneous equations

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + az &= b \end{aligned}$$

have (i) no solution (ii) a unique solution and (iii) an infinite number of solutions.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}$$

The augmented matrix

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{array}\right] \quad \text{by } R_2 - R_1,\ R_3 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{array}\right] \quad \text{by } R_3 - R_2.$$

(i) If $a - 3 = 0$ then $a = 3$ and $b \neq 10$, rank of $A = 2 <$ rank of $[A \mid b]$ so, in this case, the system has no solution.

(ii) If $a - 3 \neq 0$, then $a \neq 3$ and rank $A$ = rank $[A \mid b]$ the system has a unique solution.

(iii) If $a - 3 = 0$, then $a = 3$ and $b - 10 = 0$, rank $A$ = rank $[A \mid b] = 2 < 3$ so the system has infinite number of solutions.

## EXERCISE 4.4

1: Write each of the following system in matrix form and solve the systems.

(i)
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ 3x_2 + 2x_3 &= 1 \\ x_3 &= 2. \end{aligned}$$

(ii)
$$\begin{aligned} 3x_1 + 4x_2 &= 7 \\ 3x_2 &= 6. \end{aligned}$$

(iii)
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_2 + 2x_3 &= -2 \\ x_3 + 2x_4 &= 4 \end{aligned}$$

(iv)
$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 - x_5 &= -2 \\ x_2 - 2x_3 + x_4 - 2x_5 &= -1 \\ 2x_3 - x_4 + x_5 &= -3 \\ x_4 - 2x_5 &= 1 \end{aligned}$$

(v)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_3 + x_4 \end{aligned}$$

$$x_3 - x_4 = -3$$
$$x_3 + 2x_4 = 3.$$

**2.** Solve the system $Ax = b$, where

(i) $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 6 \\ 1 & 5 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 27 \\ 15 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 4 \\ 2 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -3 & 4 \\ 1 & -7 & -2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 6 & -2 \\ 0 & 3 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -8 \\ -3 \end{bmatrix}$

(v) $A = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

**3.** Solve the following systems,

(i)
$$2x_1 + x_2 - x_3 = 6$$
$$x_1 - 2x_2 - 2x_3 = 1$$
$$-x_1 + 12x_2 + 8x_3 = 7.$$

(ii)
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 9x_3 &= 1 \end{aligned}$$

(iii)
$$\begin{aligned} -x_1 + 2x_2 + x_3 - 2x_4 &= 2 \\ x_1 - x_2 - 2x_3 + 2x_4 &= 0 \\ -2x_1 + 4x_2 - 2x_4 &= -2. \end{aligned}$$

(iv)
$$\begin{aligned} 2x_2 - x_3 - 2x_4 &= -1 \\ -2x_1 + 2x_2 - 3x_3 - 4x_4 - 8x_5 &= 0 \\ x_3 + 3x_4 + 5x_5 &= 1 \\ x_1 + x_3 + 2x_4 + 6x_5 &= 0 \\ 2x_2 - x_3 - x_4 + 3x_5 &= -1 \end{aligned}$$

(v)
$$\begin{aligned} x_1 + 2x_2 - 2x_3 - x_4 &= 3 \\ 2x_1 + 3x_2 - 5x_3 + x_4 - 7x_5 &= 4 \\ x_2 + x_3 - x_4 + x_5 &= 2 \\ -x_1 + x_2 + 5x_3 - x_4 &= 3 \end{aligned}$$

4. For each of the following systems, find the value of $\alpha$ that makes the system $Ax = b$ consistent, for which

(i) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \\ \alpha \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & -5 \\ 6 & 7 \\ -3 & 39 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ \alpha \end{bmatrix}$

**5.** For each of the following system, find the value of $\alpha$, if any, for which the system $Ax = b$ has no solution, exactly one solution or infinitely many solutions, where,

(i) $A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ \alpha & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & 0 & 1 \\ 0 & \alpha & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix}$, $b = \mathbf{0}$

**6.** Find real $\alpha$ for which the following homogeneous system has a (i) zero solution (ii) infinite solutions.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 0 \\ \alpha x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

**7.** For what values of 'a', the following system of equations;

$$\begin{aligned} 2x - 3y + 6z - 5t &= 3 \\ y - 4z + t &= 1 \\ 4x - 5y + 8z - 9t &= a \end{aligned}$$

has (i) no solution.

(ii) Infinite number of solution.

**8.** For what values of a and b, the following system of equaitons

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + az &= b \end{aligned}$$

have (i) unique solution

(ii) no solution.

9. Determine the values of a and b for which th system

$$3x - 2y + z = b$$
$$5x - 8y + 9z = 3$$
$$2x + y + az = -1$$

(i) has a unique soluiton

(ii) has no solution

(iii) has infinity mony solution.

10. Find the values of 'a', such that, the system of equations

$$4x_1 + 9x_2 + x_3 = 0$$
$$ax_1 + 3x_2 + ax_3 = 0$$
$$x_1 + 4x_2 + 2x_3 = 0$$

11. Find the values of 'a' such that the system of equations;

$$x + ay + 3z = 0$$
$$4x + 3y + az = 0$$
$$2x + y + 2z = 0$$

has non-trivial solution.

## 4.5 DETERMINANT OF A MATRIX

The mathematical entity of a matrix has already been discussed and its uses have been exposed in the study of vector spaces. While experiencing at handling matrices one associates a real number with square matrices with entries from the real field R. It occupies a central position in the theory of matrices. The real number associated with a square matrix is called "determinant" of that matrix. This association underlines a real valued function from the set $M_n$ of all n x n square matrices with the real entries into the real field R, called the determinant function or mapping, which is denoted by det.

Thus $\det : M_n \longrightarrow R$

There are several methods of defining this "det Function". We use an inductive definition, by defining it from 1 x 1 matrix and then go on defining to n x n matrix in terms of determinant of several (n – 1) x (n – 1) matrices.

### 4.5.1 DEFINITION

Any matrix which is obtained from a given matrix by deleting certain number of rows or columns or both of the given matrix, is called a submatrix of that matrix.

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ is a submatrix of } \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \\ 1 & 0 & 8 \end{bmatrix}$$

and $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}$

Note that,

1. Any submatrix of a square matrix may not be a square matrix.
2. Any rectangular matrix can have a square submatrix.
3. Any matrix can be a submatrix of several square as well as rectangular matrices.

### 4.5.2 DEFINITION

Let M be the set of all $n \times n$ square matrices. The submatrix obtained from M by deleting ith row and jth column of M is denoted by $M_{ij}$ and $\det(M_{ij})$ is called (i, j)th minor of M.

### 4.5.3 DEFINITION

Let $M_n$ be an $n \times n$ square matrix over the field R. The "det function" from $M_n$ into R is defined by,

$$\det M_n = \begin{cases} a, & \text{if } M = [a]_{1\times 1} \\ \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} & = a_{11} \det M_{11} - a_{12} \det M_{12} \\ & \quad + \cdots\cdots + (-1)^{n+1} a_{1n} \det M_{1n} \end{cases}$$

if,

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

For example, if $M_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det M = ad - bc$

Thus determinant of a 2 x 2 matrix is obtained by cross multiplication of its entries.

Note that;

1. There are two parts of the definition of "det mapping". The first part, when M is a 1 x 1 matrix, gives that 'det mapping' is one-to-one and hence is a trivial case.

2. $\det [a] = |a|$ does not mean the absolute value of real number $a$.

3. The second part is active and non-trivial aspect of the 'det function'.

4. Evaluation of $\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + \cdots\cdots + (-1)^{n+1} a_{1n} \det M_{1n}$ is known as expansion by minors along the first row. Similarly it can be expressed by any row or column of the matrix M.
   e.g, Expansion along the ith row by minors is,
   $\det M = (-1)^{i+1} a_{i1} \det M_{i1} + (-1)^{i+2} a_{i2} \det M_{i2} + \cdots\cdots + (-1)^{i+n} a_{in} \det M_{in}$
   and expansion along the jth column by minors, is
   $\det M = (-1)^{1+i} a_{1j} \det M_{1j} + (-1)^{2+j} a_{2j} \det M_{2j} + \cdots + (-1)^{n+j} a_{nj} \det M_{nj}$

5. $(-1)^{1+j} \det M_{1j}$ is called (1, j)th cofactor of M. Similarly, $(-1)^{i+j} \det M_{ij}$ is (i, j)th cofactor of M.

6. Det A is a value of the 'det. function' at A.

### 4.5.4 EXAMPLES

1. If $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$$

2. If $C = \begin{bmatrix} 5 & 2 & -3 \\ 1 & 4 & 0 \\ 6 & 7 & -8 \end{bmatrix}$, then

$$\det C = \begin{vmatrix} 5 & 2 & -3 \\ 1 & 4 & 0 \\ 6 & 7 & -8 \end{vmatrix}$$

$$= 5\begin{vmatrix} 4 & 0 \\ 7 & -8 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 6 & -8 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 6 & 7 \end{vmatrix}$$

$$= 5(-32 - 0) - 2(-8 - 0) - 3(7 - 24)$$

$$= -160 + 16 - 3(-17)$$

$$= -160 + 16 + 51 = -93$$

3. If $A = \begin{bmatrix} -1 & 0 & 0 & -3 \\ 0 & 5 & 7 & 11 \\ 2 & 4 & 6 & -2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, then

$$\det A = \begin{vmatrix} -1 & 0 & 0 & -3 \\ 0 & 5 & 7 & 11 \\ 2 & 4 & 6 & -2 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

Expanding by the fourth row, we get.

$$= -1 \begin{vmatrix} -1 & 0 & -3 \\ 0 & 7 & 11 \\ 2 & 6 & -2 \end{vmatrix}$$

$$= -1 \left\{ -1 \begin{vmatrix} 7 & 11 \\ 6 & -2 \end{vmatrix} - 3 \begin{vmatrix} 0 & 7 \\ 2 & 6 \end{vmatrix} \right\}$$

$$= -1 \{-1 (-14 - 66) - 3 (0 - 14)\}$$

$$= -1(+ 80 + 42)$$

$$= -122$$

### 4.5.5 OBSERVATIONS FROM THE DEFINITION OF DET $M_2$

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ matrix over the real field R. Then

(i) If $r \in R$ is multiplied with the entries of any row of M and the obtained matrix is M′ then

$$\det M' = \begin{vmatrix} ar & br \\ c & d \end{vmatrix}$$

$$= ar \cdot d - br \cdot c$$

$$= r(ad - bc).$$

$$= r. \det M.$$

(ii) If M′ is obtained by interchanging any two adjacent rows of M, then

$$\det M' = \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$$= bc - ad$$

$$= -(ad - bc) = -\det M.$$

(iii) If M′ is obtained from M by adding a scalar multiple of any row to the other row, then

$$\det M' = \det \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$$

$$= (a + kc)d - c(b + kd)$$

$$= ad + kdc - cb - kcd$$

$$= ad - bc = \det M.$$

(iv) $\det M' = 1$, if M′ is a 2 x 2 identity matrix.

The observations (i) to (iv) tell us some properties of the "det function" from n x n matrices to the real values. We might expect that there is one and only one such function with these properties. We, therefore, characterize, "det function" or "determinant function" in the following theorem,

### 4.5.6 THEOREM

There is one and only one function, the "det function" under which each n x n matrix assigns to a real number and has the following four properties (i) to (iv) i.e., if A is a n x n matrix and

(i) B is obtained from A by multiplying any row of A by the real number r, then

$$\det B = r \det A.$$

(ii) B is obtained from A by interchanging two rows, then

$$\det B = (-1) \det A.$$

and $\det (R_{ij}(A)) = (-1)^{2(j-i)-1} . \det A$

(iii) B is obtained from A by adding a scalar multiple of one row to an other, then $\det B = \det A$.

(iv) $\det I_n = I$.

The properties of (i), (ii) and (iii) tell what happens to the determinant of a matrix when the elementary row operations are applied to that matrix.

The following theorem adds to some useful properties of the "det function."

### 4.5.7 THEOREM

Let M be an n x n matrix, then

(i) $\det M = 0$, if M has two equal rows.

(ii) $\det M = 0$, if M has a row of zeros.

**PROOF**

(i) Suppose two of the rows of M are equal and M′ is the matrix obtained from M by interchanging two equal rows of M. Then by property (ii) of theorem 4 · 5 · 6.

$$\det M' = -\det M.$$

As two equal rows of M are interchanged the matrix remains unchanged, therefore $M' = M$ and $\det M' = \det M$.

Thus $\det M = -\det M$

and $2 \det M = 0$

which implies that $\det M = 0$.

(ii) Suppose that M has a row of zeros'. Choose any other row of M and add it to the row of zeros to obtain a matrix M′. By the property (iii) of theorem [4.5.6], $\det M' = \det M$.

But M′ has two equal rows, therefore $\det M' = 0$ by (i).

Hence $\det M = 0$

For example,

(i) If $M = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 5 & 4 & 3 \\ -7 & 6 & 5 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix}$ then $\det M = 0$, because

$R_1$ and $R_4$ are equal rows of M.

(ii) If $M = \begin{bmatrix} 3 & 2 & -8 & 7 \\ -1 & 5 & 6 & 3 \\ 0 & 0 & 0 & 0 \\ 8 & 9 & 8 & 4 \end{bmatrix}$, then det M = 0,

because $R_3$ of M contains all the zeros.

### 4.5.8 THEOREM

Let M be an n x n square matrix such that each entry $a_i$ of ith row of M is sum of two real number as $a_i = b_i + c_i$ then

$$\det M = \det M_1 + \det M_2 \qquad 1 \le i \le n$$

where $M_1$ and $M_2$ are the matrices obtained from M by replacing the ith row of M by $(b_i)$ and $(c_i)$ respectively.

**PROOF** det M

$$= (-1)^{i+1} a_{i1} \det M_{i1} + (-1)^{i+2} a_{i2} \det M_{i2}$$

$$+ \ldots\ldots\ldots + (-1)^{i+n} a_{in} \det M_{in}$$

But $a_{ij} = b_{ij} + c_{ij}$ , $1 \le j \le n$, therefore,

$$\det M = (-1)^{i+1} (b_{i1} + c_{i1}) \det M_{i1} + (-1)^{i+2} (b_{i2} + c_{i2}) \det M_{i2}$$

$$+ \ldots\ldots\ldots + (-1)^{i+n} (b_{in} + c_{in}) \det M_{in}$$

$$= (-1)^{i+1} b_{i1} \det M_{i1} + (-1)^{i+1} c_{i1} \det M_{i1}$$

$$+ (-1)^{i+2} b_{i2} \det M_{i2} + (-1)^{i+2} c_{i2} \det M_{i2}$$

$$+ \ldots\ldots\ldots + (-1)^{i+n} b_{in} \det M_{in} + (-1)^{i+n} c_{in} \det M_{in}.$$

Re-arranging the terms, we get,

$$= \{(-1)^{i+1} b_{i1} \det M_{i1} + (-1)^{i+2} b_{i2} \det M_{i2} + \ldots + (-1)^{i+n} b_{in} \det M_{in}\}$$

$$+ \{(-1)^{i+1} c_{i1} \det M_{i1} + (-1)^{i+2} c_{i2} \det M_{i2} + \ldots + (-1)^{i+n} c_{in} \det M_{in}\}$$

$$= \det M_1 + \det M_2$$

Same result holds if same property holds in any of the columns of M.

**4.5.9 THEOREM**

If $M = \begin{bmatrix} a_{11} & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_{22} & 0 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & a_{ii}\ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & a_{nn} \end{bmatrix}$ is diagonal n x n matrix then

$$\det M = a_{11} \cdot a_{22} \ldots\ldots a_{nn}$$

i.e. det M is equal to the product of all the diagonal entries of M.

**PROOF**

$$\det M = \begin{vmatrix} a_{11} & 0 & \ldots\ldots & 0 \\ 0 & a_{22} & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & a_{ii} & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots\ldots & a_{nn} \end{vmatrix}$$

$$= a_{11} \det \begin{bmatrix} 1 & 0 & \ldots\ldots & 0 \\ 0 & a_{22} & \ldots\ldots & 0 \\ :::::: & :::::: & :::::: & :: \\ 0 & 0 & \ldots\ldots & a_{nn} \end{bmatrix}, \text{ by (i) of}$$

Theorem [4.5.6]. By the repeated application of this property, one gets that

$$\det M = a_{11} \cdot a_{22} \ldots\ldots a_{nn} \cdot \det I_n.$$

$$= a_{11} \cdot a_{22} \ldots\ldots a_{nn} \cdot 1, \text{ by property (iv)}$$

$$= a_{11} \cdot a_{22} \ldots\ldots a_{nn}.$$

### 4.5.10 THEOREM

The determinant of an upper triangular matrix M is equal to the product of its diagonal entries.

**PROOF**

Let $M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be an $n \times n$ matrix. Then

$$\det M = a_{nn} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

by using property (i) of theorem 4.5.6 on $R_n$.

By applying property (iii) further by $(R_{n-1} - a_{n-1,n} R_n)$, we get,

$$\det M = a_{nn} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Apply (i) on $R_{n-1}$, and then property (iii) on $R_1, R_2, \ldots R_{n-2}$, we get that,

$$\det M = a_{nn} a_{n-1,n-1} \det \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & 0 \\ 0 & a_{22} & \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\ 1 & \cdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By applying repeatedly, the same process, one gets eventually that

$$\det M = a_{nn} \cdot a_{n-1\,n-1}\, a_{n-2,n-2} \ldots\ldots\ldots\ldots a_{11}.\ \det I_n.$$

$$= a_{nn}\, a_{n-1\,n-1} \ldots\ldots\ldots a_{11}.1$$

$$= a_{nn}\, a_{n-1\,n-1} \ldots\ldots\ldots a_{11}$$

**NOTE**

1. To compute the determinant of any square matrix, we need only transform it into an upper triangular matrix. We do not need to transform a matrix all the way to its echelon form in order to find the determinant of that matrix.

2. If any of the diagonal entries of M in its triangular form is zero then det M = 0

**EXAMPLE 1**

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{bmatrix}, \text{ then}$$

$$\det A = (1)(-1)(7) = -7$$

**EXAMPLE 2**

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 8 & 7 \\ 4 & 1 & -3 \end{bmatrix}, \text{ then}$$

$$\det A = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & -7 & -15 \end{bmatrix}, \text{ by } R_2 - 3R_1,\ R_3 - 4R_1$$

$$= \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & -22 \end{bmatrix}, \text{ by } R_3 + \frac{7}{2} R_2.$$

$$= (1)(2)(-22) = -44$$

Thus, to evaluate det A, if A is an n x n square matrix,

1. Use row operations to transform A into an upper triangular matrix.
2. Then use the fact that the determinant of an upper triangular matrix is equal to the product of its diagonal entries.
3. This method is usually the simplest method for computing det A, if A is an n x n matrix whenever $n > 2$.
4. Sometimes it is more efficient a technique to combine these two methods of evaluating a det M, that is, by minors and by the use of row operations.

**FOR EXAMPLE**

$$\text{If } A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 3 & 2 \\ 1 & -1 & 2 & 1 \end{bmatrix}, \text{ then}$$

$$\det A = \det \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \text{ by row operations } R_3 + R_1,\ R_4 - R_1$$

Expanding the transformed matrix by the first column, we get it

$$= \det \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}, \quad \text{by } R_2 - 2R_1$$

Again expanding by the first column,

$$= \det \begin{bmatrix} -1 & -5 \\ 2 & 0 \end{bmatrix} = (-1)(0) - (2)(-5)$$

$$= 10$$

### 4.5.11 THEOREM

Let M be a square matrix. Then

$$\det M = \det M^t,$$

where $M^t$ is a matrix transpose of M.

**PROOF** Notice that the row vectors of $M^t$ are the column vectors of M and the column vectors of $M^t$ are the row vectors of M. If M is a square matrix than $M^t$ is also a square matrix. For the expansion of det M by minors along the ith row, let $a_{ij}$ denote the (i, j)th entry of M and det $M_{ij}$ denote the (i, j)-minor of M. Then (i, j)th entry of $M^t$ is $(a_{ji})$ and the (i, j) – minor of $M^t$ is det $(M_{ji})^t$. Using the formula for the expansion of det $M^t$ by minons along the ith column, we obtain the formula for the expansion of det M along the ith row, as,

$$\det M^t = (-1)^{1+i} a_{1i} \det M_{1i}^t + (-1)^{2+i} a_{2i} \det M_{2i}^t + \ldots + (-1)^{n+i} a_{ni} \det M_{ni}^t$$

$$\text{or } \det M = (-1)^{i+1} a_{i1} \det M_{i1} + (-1)^{i+2} a_{i2} \det M_{i2} + \ldots + (-1)^{i+n} a_{in} \det M_{in}$$

**COR. 1** We may operate row x row multiplication of the determinants induced from the row x column multiplication of the matrices.

For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}^2 = \begin{vmatrix} 14 & 8 & 4 \\ 8 & 5 & 2 \\ 4 & 2 & 2 \end{vmatrix}$$

Noted that square of a determinant is symmetric.

**COR 2** $\det(E_{ij} \cdot A) = \det E_{ij} \cdot \det A.$

## 4.5.12 DETERMINANT OF A PRODUCT MATRIX

We know that if M is a n x m matrix and E be an elementary matrix then

(i) Multiplying M on the left by E has the same effect as applying the respective row operation. The row operation that transforms M into EM transforms I into E.

(ii) If M is a non-singular matrix then M can be factored into a product of elementary matrices.

i.e. If $E_k \cdot E_{k-1} \ldots\ldots\ldots E_2 \cdot E_1 M = I$
where $E_1, E_2 \ldots\ldots, E_k$ are the elementary matrices corresponding to the row operations $E_1, E_2, \ldots\ldots, E_k$, that transform M to I, then

$$M = E_1^{-1} E_2^{-1} \ldots\ldots\ldots E_k^{-1}$$

as inverses of elementary matrices are elementary matrices, therefore so in their product.

## 4.5.13 THEOREM

If M and N are two n x n square matrices then

$$\det (M \cdot N) = \det (M) \cdot \det (N).$$

**PROOF** If the elementary matrix E is obtained from $I_n$ by multiplying a row of $I_n$ by c, then EM is obtained from M by multiplying a row of M by c.
Hence $\det (EM) = c \det M$, and $\det E = c$

So $\det (EM) = \det (E) . \det (M).$

Suppose E is obtained from $I_n$ by interchanging two rows. Then EM is obtained from M by interchanging two rows and

$$\det (EM) = -\det M, \quad \text{where } \det E = -1$$
$$= (-1) \det M$$

$$= \det E.\ \det M$$

If E is obtained from $I_n$ by adding a scalar multiple of one row to an other, then EM is obtained from M by adding a scalar multiple of one row to another and

$$\det EM = \det M, \text{ where } \det E = 1$$
$$= 1.\ \det M$$
$$= \det E.\ \det M$$

Thus we have shown that $\det (MN) = \det M \cdot \det N$. if M is an elementary matrix.

If M is non-singular then M can be factored as a product of elementary matrices therefore,

$$\det (M \cdot N) = \det M \cdot \det N.$$

If M is singular then $\det M = 0$, so $\det (MN) = \det M.\ \det N$.

Hence it completes the proof of this product formula.

**COR. 1** If A is non-singular, then $\det A \neq 0$ and $\det A^{-1} = \dfrac{1}{\det A}$.

**PROOF** $1 = \det I = \det (A \cdot A^{-1}) = \det A.\ \det A^{-1}$

$$\Rightarrow \quad \det A^{-1} = \frac{1}{\det A}.$$

If $\det A = 1$ then $\det A^{-1} = 1$

**COR. 2** $\det (M.M^t) = \det M.\ \det M^t = \det M \det M$
$$= (\det M)^2$$

The following theorem characterizes the non-singular matrices by their determinants.

### 4.5.14 THEOREM

Let A be a square matrix. Then A is non-singular if and only if $\det A \neq 0$.

**PROOF** We know that a non-singular square matrix has an inverse. Thus it is invertible. Let A be an n x n matrix and M be the echelon form of A. Then A can be transformed into M by a sequence of elementary row operations. Each row operation either leaves the determinant unchanged or multiplies the determinant by a non-zero real number. Thus $\det A \neq 0$ if and only if $\det M \neq 0$.

If the matrix M is upper triangular, so its determinant is non-zero if and only if all of its diagonal entries are non-zero. Since M is the reduced echelon matrix, then it means that M contains n pivot element 1's. But the number of 1's is equal to the rank of A. Therefore $\det M \neq 0$ if and only if rank of A is n.

Thus it concludes that $\det A \neq 0$ if and only if rank of A is n. We have proved then, that A is non-singular.

## 4.5.15 FORMULA FOR INVERSE OF A MATRIX

$$\text{Let } M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ \vdots\vdots & \vdots\vdots & \vdots\vdots & \vdots\vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots & a_{n,n} \end{bmatrix}$$

Expanding det M by the ith row, we have

$$\det M = (-1)^{i+1} a_{i1} \det M_{i1} + (-1)^{i+2} a_{i2} \det M_{i2} + \ldots\ldots + (-1)^{i+n} a_{in} \det M_{in}$$

If cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$ then

$$\det M = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots\ldots\ldots + a_{in} C_{in}$$

det M is the dot product of the ith row vector of M and the ith row vector of

$$M' = \begin{bmatrix} C_{11} & C_{12} & \ldots\ldots & C_{1n} \\ C_{21} & C_{22} & \ldots\ldots & C_{2n} \\ \vdots\vdots & \vdots\vdots & \vdots\vdots & \vdots \\ C_{n1} & C_{n2} & \ldots\ldots & C_{nn} \end{bmatrix}$$

which is equal to the dot product of ith row vector of M and ith column vector $(M')^t$, which is the same as the (i, j)th entry in the matrix

$$M \cdot (M')^t.$$

The diagonal entries in this product are all equal to det M and the entries away from the diagonal are all zero.

Hence $M \cdot (M')^t = \det M \cdot I_n$.

If M is non-singular then det $M \neq 0$ and

$$\frac{1}{\det M} \cdot (M) \cdot (M')^t = I_n.$$

or $$M \cdot \frac{(M')^t}{\det M} = I_n.$$

It follows that $\frac{(M^c)^t}{\det M}$ is the inverse of M.

From the discussion one is led to the following theorem,

### 4.5.17 THEOREM

Let M be an invertible matrix. Then

$$M^{-1} = \frac{(\text{Adj } M)}{\det M}$$

Where Adj M is the matrix obtained from M by replacing each entry $a_{ij}$ of M by its cofactor $(-1)^{i+1} \cdot \det M_{ij}$ and then finding its transpose i.e the adjoint matrix of M is the transpose of the co-factor matrix so that Adj $M = (M^c)^t$

This method is convenient upto a 3 x 3 square matrix. For matrices of higher dimension it becomes difficult to handle.

**EXAMPLE 1** Find inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \text{ by the use of cofactors of A.}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1\,(-1) = 1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = +1\,(-1) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = -(3) = -3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1\,(2-1) = 1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -(-3) = 3$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = 1(-1) = -1$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = -1(1) = -1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 1(2+3) = 5$$

Thus cofactor matrix $A^c =$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 3 \\ -1 & -1 & 5 \end{bmatrix} \text{ and } (\text{Adj } A) = (A^c)^t = \begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}$$

$$\det A = (2,3,1)\cdot(1,1,-1) = 2\cdot 1 + 3\cdot 1 + 1(-1)$$

$$= 2 + 3 - 1 = 4$$

$$\text{Therefore } A^{-1} = \begin{bmatrix} 1/4 & -3/4 & -1/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & 5/4 \end{bmatrix}$$

The student is advised to verify that

$$A\cdot A^{-1} = I_3 = A^{-1}\cdot A.$$

**EXAMPLE 2** Without expanding show that

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \det\begin{bmatrix} e & b & h \\ d & a & g \\ f & c & k \end{bmatrix}$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = (-1)\det\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & k \end{bmatrix}, \text{ by interchanging } R_1 \text{ and } R_2$$

$$= (-1)\det\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & k \end{bmatrix}^t, \quad \det A = \det A^t.$$

$$= (-1)\det\begin{bmatrix} d & a & g \\ e & b & h \\ f & c & k \end{bmatrix},$$

$$= (-1)(-1)\det\begin{bmatrix} e & b & h \\ d & a & g \\ f & c & k \end{bmatrix}, \text{ by interchanging } R_1 \text{ and } R_2$$

$$= \det\begin{bmatrix} e & b & h \\ d & a & g \\ f & c & k \end{bmatrix}$$

**EXAMPLE 3** If $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, then prove, without expanding, that $\det A = 0$.

$\det A = \det A^t = \det(EA)$, where $E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is elementary matrix.

$= \det E \cdot \det A.$
$= (-1) \det A.$
$= -\det A.$

Which gives that $2 \det A = 0 \Rightarrow \det A = 0$

**EXAMPLE 4** Find the value of the determinant

$$\det A = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

By $C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} = (x+3a) \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$$= (x+3a) \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} \text{ by, } R_2 - R_1, R_3 - R_1, R_4 - R_1$$

$= (x+3a) \cdot 1 \cdot (x-a)(x-a)(x-a)$, where the matrix is triangular.

$= (x+3a)(x-a)^3$.

**EXAMPLE 5** If $A = \begin{bmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}$, then find det A.

$$\det A = \det \begin{bmatrix} a^3-1 & 3a^2-3 & 3a-3 & 0 \\ a^2-1 & a^2+2a-3 & 2a+2 & 0 \\ a-1 & 2a-2 & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \text{ by } R_3-R_4, R_2-R_4, R_3-R_4$$

$$= \det \begin{bmatrix} a^3-1 & 3(a^2-1) & 3(a-1) \\ a^2-1 & (a+3)(a-1) & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{bmatrix}, \text{by expanding along 4th column}$$

$$= (a-1)^3 \det \begin{bmatrix} a^2+a+1 & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= (a-1)^3 \det \begin{bmatrix} a^2 & 2a & 1 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \text{ by } R_1 - R_2$$

$$= (a-1)^3 \det \begin{bmatrix} a^2-1 & 2(a-1) & 0 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \text{ by } R_1 - R_3.$$

$$= (a-1)^4 \det \begin{bmatrix} a+1 & 2 & 0 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= (a-1)^4 \det \begin{bmatrix} a+1 & 2 & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \text{ by } R_2 - 2R_3$$

$$= (a-1)^5 \det \begin{bmatrix} a+1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= (a-1)^5 \det \begin{bmatrix} a+1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= (a-1)^5 (a+1-2)$$

$$= (a-1)^6$$

**EXAMPLE 6** For which value of $\lambda$, the det A = 0, for

$$A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda-3 & -2 \\ 2 & 1 & \lambda+2 \end{bmatrix}$$

Expansion along the first row,

$$\det A = \lambda \det\begin{bmatrix} \lambda-3 & -2 \\ 1 & \lambda+2 \end{bmatrix} + 1 \det\begin{bmatrix} 0 & -2 \\ 2 & \lambda+2 \end{bmatrix}$$

$$= \lambda\{\lambda - 3)(\lambda + 2) + 2\} + 1(4)$$

$$= \lambda(\lambda^2 - \lambda - 6 + 2) + 4$$

$$= \lambda^3 - \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 1, 2, -2.$$

If $\lambda \neq 1, 2, -2$, then A is non-singular, i.e. $\det A \neq 0$.

If $\lambda = 1, 2$ or $-2$ then A is singular, i.e $\det A = 0$.

**EXAMPLE 7** Without expansion, show that

$$\det\begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix} = 0$$

$$\det\begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix}$$

$$= \det\begin{bmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix} \text{ by } R_1 + R_2 + R_3$$

$$= 0, \text{ because of } R_1 \text{ to be zero.}$$

**EXAMPLE 8** Verify that,

$$\det\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (x - y)(y - z)(z - x)$$

$$\det\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix} = \det\begin{bmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{bmatrix} \text{ by } R_2 - R_1,\ R_3 - R_1$$

$$= (y-x)(z-x)\det\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 1 & z+x\end{bmatrix}$$

$$= (y-x)(z-x)\det\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 0 & z-y\end{bmatrix}, \text{ by } R_3 - R_2$$

$$= (y-x)(z-x)\cdot(z-y) = (x-y)(y-z)(z-x).$$

## EXERCISE 4.5

1. Calculate the values of the det function for the matrices,

(i) $\begin{bmatrix}-2 & 2 & 9\\ 3 & -4 & 1\\ 5 & 0 & 8\end{bmatrix}$ (ii) $\begin{bmatrix}0 & 2 & -3\\ -2 & 0 & 3\\ 2 & -3 & 0\end{bmatrix}$ (iii) $\begin{bmatrix}1/3 & 0 & 1/3\\ 1/2 & 1 & 1/2\\ -1 & 1/3 & 2/3\end{bmatrix}$

(iv) $\begin{bmatrix}1 & -1 & 1 & 0\\ -1 & 1 & 0 & 1\\ 1 & 0 & 1 & -1\\ 0 & 1 & -1 & 1\end{bmatrix}$ (v) $\begin{bmatrix}7 & 0 & 2 & 1\\ 2 & -2 & 1 & 0\\ 0 & 1 & 0 & -2\\ 5 & 1 & -1 & 3\end{bmatrix}$

2. If $A = \begin{bmatrix}2 & 1 & 0 & 5\\ -2 & 0 & -1 & 1\\ 3 & 4 & 3 & 0\\ 1 & -2 & 0 & 6\end{bmatrix}$, then calculate $|A|$.

(i) By expanding along the third row.

(ii) By expanding along the third column.

(iii) By expanding along the second row.

**3.** Determine all the values of $\lambda$ for which $\det A = 0$

$$\text{for } A = \begin{bmatrix} -\lambda & 0 & -1 \\ -2 & 2-\lambda & -3 \\ -1 & 2 & -2-\lambda \end{bmatrix}$$

**4.** For each real number x, let f(x) be defined by the formula

$$f(x) = \det \begin{bmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Without expanding, find two values of x for which $f(x) = 0$

**NOTE.** The replacement $\alpha$ for x which makes $f(\alpha) = 0$ implies that $(x - \alpha)$ is a factor of f(x).

**5.** Explain why; if A has one row which is scalar multiple of another row, then

$\det A = 0.$

**6.** Calculate

(i) $\begin{vmatrix} 2 & 0 & 0 & -3 & 0 \\ 3 & 4 & -2 & 1 & 5 \\ -3 & -1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{vmatrix}$ (ii) $\begin{vmatrix} -1 & 2 & 3 & -1 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 3 & -2 & 1 & 2 \\ 0 & 2 & -1 & 3 & 1 \\ -1 & 0 & 2 & 0 & -4 \end{vmatrix}$

**7.** Evaluate

(i) $\det \begin{bmatrix} 2 & 3 & -1 & 4 \\ 6 & -1 & 2 & 0 \\ 3 & 5 & 7 & -2 \\ 1 & 1 & 3 & 4 \end{bmatrix}$ (ii) $\det \begin{bmatrix} 2 & 7 & 1 & 8 \\ 2 & 8 & 1 & 8 \\ 2 & 8 & 4 & 5 \\ 9 & 0 & 4 & 5 \end{bmatrix}$

(iii) $\det\begin{bmatrix} 1 & 2 & 1 \\ x & 3 & 3 \\ x^2 & -1 & 2 \end{bmatrix}$, for $x = -1$ (iv) $\det\begin{bmatrix} x-1 & 1 & 3 \\ 2 & x-3 & 1 \\ -2 & 4 & x+1 \end{bmatrix}$, for $x = 2$.

8. Answer the following statements true or false? Support your answer with proof, if A is an n x n matrix and;

(a) If A has one row that is a scalar multiple of another, then det A = 0.

(b) If A has two columns that are equal, then det A = 0.

(c) If det A = 0, then the echelon matrix of A has a row of zeros.

(d) If $\det A^3 = 0$, then det A = 0.

(e) If $\det A^3 = 1$, then det A = 1.

(f) If $A^3 = A$, then det A = 0 or det A = 1

(g) If A is upper triangular and has determinant equal to 0, then atleast one of the diagonal entries in A must be zero.

(h) det 2A = det A.

9. If A is a 3 x 3 matrix with det A = 2; then find

(i) $\det A^5$ (ii) det 5A (iii) det (–A)

(iv) $\det A^{-1}$ (v) $\det A^t$.

10. Show, without expanding, that det A = 0, for

(i) $A = \begin{bmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{bmatrix}$

11. Compute |A|, by expressing $A = \begin{bmatrix} 1 & 5 & 0 & 3 & -4 \\ 2 & 3 & 7 & -1 & 6 \\ -3 & -10 & 2 & 4 & 3 \\ 0 & -2 & 3 & 1 & 5 \\ 4 & 15 & 2 & 13 & -9 \end{bmatrix}$

into its upper triangular form.

**18.** If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and A, B, C etc. are the cofactors of $a$, b, c etc. respectively. Then show that

(i) $BC - F^2 = a\Delta$

(ii) $CA - G^2 = b\Delta$

(iii) $AB - H^2 = c\Delta$

(iv) $aG + hF + gC = 0$

(v) $hG + bF + fC = 0$

(vi) $gG + fF + cC = \Delta$

**19.** Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where $A_1, B_1, C_1, \ldots\ldots$ are the cofactors of $a_1, b_1, c_1 \ldots\ldots$ in the determinant on the left.

**20.** Prove that,

(i) $$\begin{vmatrix} a+b & b+c & c+a \\ p+q & q+r & r+p \\ l+m & m+n & n+l \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ l & m & n \end{vmatrix}$$

(ii) $$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

**21.** Show that,

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$ (ii) $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

$= 4(b+c)(c+a)(a+b).$

**22.** Show that,

$$\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

**23.** Show that,

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

**24.** Show that,

$$\begin{vmatrix} (x-a)^2 & (y-a)^2 & (z-a)^2 \\ (x-b)^2 & (y-b)^2 & (z-b)^2 \\ (x-c)^2 & (y-c)^2 & (z-c)^2 \end{vmatrix} = 2 \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \cdot \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

# 5

# SEQUENCES AND SERIES

## 5.1 SEQUENCES

According to dictionary meaning, a sequence is, "the following of one thing after another." In mathematics we define a sequence as a succession of numbers that never terminates. The numbers in the sequence are called the terms of the sequence. In a sequence there is one term for each positive integer.

**EXAMPLE 1** Consider the sequence

$$\underset{\substack{\uparrow\\ 1\text{st}\\ \text{term}}}{\frac{1}{2}}, \underset{\substack{\uparrow\\ 2\text{nd}\\ \text{term}}}{\frac{1}{4}}, \underset{\substack{\uparrow\\ 3\text{rd}\\ \text{term}}}{\frac{1}{8}}, \underset{\substack{\uparrow\\ 4\text{th}\\ \text{term}}}{\frac{1}{16}}, \underset{\substack{\uparrow\\ 5\text{th}\\ \text{term}}}{\frac{1}{32}}, \cdots, \underset{\substack{\uparrow\\ n\text{th}\\ \text{term}}}{\frac{1}{2^n}}, \cdots\cdots$$

It can be observed there is one term for each positive integer.

The terms of this sequence form an infinite set of real numbers, which can be written as

$$A = \left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n}, \cdots \right\}$$

i.e. The set A has elements of the form $\frac{1}{2^n}$, where n is positive integer.

An alternative to describe this set A is that we define the function f by the rule $f(n) = \frac{1}{2^n}$, where the domain of function is the set of positive integers. Then the set A is exactly the set of values of the the function f. i.e.,

$$f(1) = \frac{1}{2}, \; f(2) = \frac{1}{2^2}, \; f(3) = \frac{1}{2^3}, \ldots, f(n) = \frac{1}{2^n}, \cdots$$

In general we have the following definition.

### 5.1.1 DEFINITION (SEQUENCE)

A sequence is a function whose domain is the set of positive integers.

The values taken by the function are called terms of the sequence.

**NOTATIONS** Instead of f(n), terms of the sequence are denoted by $a_n$ where n = 1, 2, 3, .....

and sequence is usually denoted by the symbol $\{a_n\}$. Thus $\{a_n\}$ is equivalent to

$$\begin{array}{ccccccccc} a_1 & , & a_2 & , & a_3 & , \ldots , & a_n & , & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & , & 2 & , & 3 & , \ldots & n & , & \cdots \end{array}$$

elements in the range

elements in the domain

**EXAMPLE 2** The following are sequences of real numbers.

(i) $\{a_n\} = \left\{\frac{1}{n}\right\}$ (ii) $\{a_n\} = \left\{\sqrt{n}\right\}$

(iii) $\{a_n\} = \{\sin n\}$ (iv) $\{a_n\} = \left\{(-1)^n\right\}$

### 5.1.2 CONVERGENT SEQUENCES

For the sequence $\left\{\frac{1}{n}\right\}$ we see as $n \to \infty$, $\frac{1}{n} \to 0$.

We say that the term $\frac{1}{n}$ approach the limit zero and that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0. In contrast, the terms of the sequences $\left\{\sqrt{n}\right\}$ and $\left\{(-1)^n\right\}$ do not approach a limit as $n \to \infty$.

We have the following definition.

### 5.1.3 DEFINITION

A sequence $\{a_n\}$ is said to converge to a number L if for every $\epsilon > 0$ there exists a positive number N such that

$$|a_n - L| < \epsilon \text{ whenever } n > N$$

i.e. $\lim_{n \to \infty} a_n = L.$

When $\lim_{n \to \infty} a_n$ does not exist, then the sequence is called divergent sequence. If $\{a_n\}$ converges, then all but a finite number of the terms $a_n$ are in the interval

$$]L - \in, L + \in [, \text{ for } n > N.$$

**EXAMPLE 1** Using this definition prove that $\left\{ \frac{1}{\sqrt{n}} \right\}$ converges to 0.

**SOLUTION** Let $\in = 0.01 > 0$ be given.

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < 0.01$$

$$\Rightarrow \quad \frac{1}{\sqrt{n}} < 0.01$$

$$\Rightarrow \quad \sqrt{n} > [0.01]^{-1}$$

$$\Rightarrow \quad n > (0.01)^{-2}$$

$$n > \frac{1}{(0.01)^2} = \frac{1}{0.0001}$$

$$n > \frac{1}{\frac{1}{10000}}$$

or $n > 10{,}000$

So here we have calculated

$N = 10{,}000$ in this case.

i.e. N depends upon $\in$.

But in practice we take limit of

$$a_n \text{ as } n \to \infty$$

If $\lim_{n \to \infty} a_n$ is finite, definite and unique we say $\{a_n\}$ is convergent otherwise divergent.

**EXAMPLE 2** $\{n^2 + n\}$ is divergent because $\lim_{n \to \infty} (n^2 + n) = \infty$.

**EXAMPLE 3** $\{8\}$ is convergent sequence.

A constant sequence is always convergent.

**EXAMPLE 4** Show that the sequence $\left\{(n + 1)^{1/n}\right\}$ converges to 1.

**SOLUTION** If $y = (n + 1)^{1/n}$

Using L' Hôpital's rule, $\lim_{n \to \infty} \log_e y = \lim_{n \to \infty} \left[\frac{\log(n+1)}{n}\right] = 0$

Consequently $\lim_{n \to \infty} (n + 1)^{1/n} = e^0 = 1$

### 5.1.5 THEOREM

(i) The sequence $\{r^n\}$ converges to 0 if $|r| < 1$

(ii) The sequence $\{r^n\}$ diverges if $|r| > 1$

(iii) For any positive rational number r, $\left\{\frac{1}{n^r}\right\}$ converges to zero.

**PROOF**

(i) Let $\epsilon = 0.001$

$|r^n - 0| < \epsilon$

$-\epsilon < r^n < \epsilon$

Considering $r^n < \epsilon$

$$n \log r < \log \epsilon, \{\text{suppose } r > 0\}$$

$$n > \frac{\log \epsilon}{\log r}, \{\log r < 0\}$$

$$\therefore \quad n > \frac{\log \epsilon}{\log r} = N(\text{say}).$$

$\therefore$ $\{r^n\}$ is convergent to 0.

Alternatively $\lim_{n \to \infty} r^n = 0$ if $|r| < 1$.

(ii) $\lim_{n \to \infty} r^n = \pm \infty$ if $|r| > 1$, $\therefore$ divergent

(iii) $\lim_{n \to \infty} \frac{1}{n^r} = 0$, if r is +ve rational number. Therefore convergent.

### 5.1.6 REMARK

In 1772 the German Astronomer Johann Elert Bode studied the sequence

$$0, 3, 6, 12, 24, 48, 96, \ldots$$

By adding 4 to each term and dividing the result by 10, he obtained the sequence

$$0.4, 0.7, 1.0, 1.6, 2.8, 5.2, 10, \ldots\ldots$$

If the number 1 represents an Astronomical unit of length which is equal to ninety-three million miles or the mean distance from the earth to sun, then 0.4, 0.7, 1.0, 1.6, 2.8, 5.2, and 10 predict fairly accurately the distances of the planets Mercury, Venus, Earth, Mars, Jupiter, and Saturn from the sun.

## EXERCISE 5.1

Q.1 Find the $\lim_{n \to \infty} a_n$, if it exists, of each sequence having the given general term $a_n$.

(i) $a_n = 20(-1)^{n+1}$ (ii) $a_n = ne^{-n}$

(iii) $a_n = \cos \pi n$ (iv) $a_n = n \sin 6/n$

(v) $a_n = \log_e\left(\frac{4n+1}{3n-1}\right)$ (vi) $a_n = \sqrt{n+1} - \sqrt{n}$

(vii) $a_n = \log n / \log 3n$

Q.2 How many pairs of rabbits will be produced in a year begining with a single pair, if in every month each pair bears a new pair which become productive from the second month on?

Q.3. A regular n-sided polygon is circumscribed by a circle of radius r. Show that the area of the polygon is given by the formulae.

$$A_n = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right).$$

and then show that sequence $\{A_n\}$ converges to $\pi r^2$.

## 5.2.1 MONOTONIC SEQUENCES

It is not always easy nor even possible to determine whether a sequence $\{a_n\}$ converges by seeking the exact value of $\lim_{n \to \infty} a_n$. In order to achieve the purpose we need to define Monotonic sequences.

### DEFINITION

A sequence is said to be monotonic if it is one of the following type :-

(i) $a_1 > a_2 > a_3 > a_4 > \cdots > a_n > a_{n+1} > \cdots\cdots$

(ii) $a_1 < a_2 < a_3 < \cdots < a_n < a_{n+1} < \cdots\cdots$

(iii) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots\cdots$

(iv) $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots\cdots$

(i) and (iii) are Monotonically decreasing while (ii) and (iv) are Monotonically increasing sequences.

The following table is an aid to observe monotonically of a sequence very accurately.

| Form | If | Then |
|---|---|---|
| A function f(x) such that $f(n) = a_n$ | $f'(x) > 0$ for all x<br>$f'(x) < 0$ for all x | $\{a_n\}$ is increasing<br>$\{a_n\}$ is decreasing |
| The ratio $\frac{a_{n+1}}{a_n}$<br>where $a_n > 0$ for all n | $\frac{a_{n+1}}{a_n} > 1$ for all n<br>$\frac{a_{n+1}}{a_n} < 1$ for all n | $\{a_n\}$ is increasing<br>$\{a_n\}$ is decreasing |
| The difference $a_{n+1} - a_n$ | $a_{n+1} - a_n > 0$ for all n<br>$a_{n+1} - a_n < 0$ for all n | $\{a_n\}$ is increasing<br>$\{a_n\}$ is decreasing |

**EXAMPLE** Show that $\left\{\frac{n}{e^n}\right\}$ is a decreasing sequence.

**SOLUTION** If we define $f(x) = \frac{x}{e^x}$, then $f(n) = a_n$,

$$f'(x) = \frac{e^x - xe^x}{(e^x)^2} = \frac{1 - x}{e^x} < 0, \text{ for } x > 1$$

so f(x) is decreasing.

Thus the sequence is decreasing.

## 5.2.2 BOUNDED SEQUENCE

A sequence $\{a_n\}$ is said to be bounded if there exists a positive number B, such that,

$$|a_n| \leq B \text{ for all } n.$$

EXAMPLE The sequence $\left\{\frac{3n+1}{n+1}\right\}$ is bounded above by 3. Because

$$\left|\frac{3n+1}{n+1}\right| \leq 3, \forall n \in N,$$

N being the set of Natural numbers.

## 5.2.3 THEOREM

A bounded Monotonic sequence is convergent.

PROOF $\{a_n\}$ is bounded

$$\therefore |a_n| \leq B, \forall n \in N.$$

$\{a_1, a_2, a_3, \ldots\ldots\}$ is bounded above.

Let sequence actually converge to L.

For $\epsilon > 0$, $L - \epsilon < L$

As result $L - \epsilon$ is not the least upper bound. Hence there exists a positive integer N such that

$$a_N > L - \epsilon.$$

But $\{a_n\}$ is increasing sequence.

$$\therefore L - \epsilon \leq a_N \leq a_{N+1} \leq a_{N+2} \leq \ldots \leq L + \epsilon.$$

So the conclusion is,

for all $n > N$

$$L - \epsilon \leq a_n \leq L + \epsilon$$

or $$|a_n - L| < \epsilon.$$

So $\lim_{n \to \infty} a_n = L.$

$\therefore$ sequence is convergent.

**EXAMPLE** Show that the sequence

$$\left\{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right\} \text{ converges.}$$

**SOLUTION** The ratio, $\frac{a_{n+1}}{a_n}$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)} \times \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$= \frac{2n+1}{2n+2} = 1 - \frac{1}{2(n+1)} < 1$$

So the sequence is Monotonic.

Now $a_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < 1$

Because in each fraction Numerator is less than denominator.

So it is bounded. i.e.

$$0 < a_n < 1, \quad \text{for all n.}$$

$$|a_n| < 1.$$

Hence it is convergent.

**NOTE** Every convergent sequence is bounded. But it is not always true that every bounded sequence is convergent. e.g. $\{(-1)^n\}$

$$|a_n| \leq 1, \forall n \in N.$$

but $\{(-1)^n\}$ is not convergent.

But on the other hand if a sequence is not bounded, then it is necessarily divergent.

e.g. $\{2n + 3\}$.

## EXERCISE 5.2

Show that the following sequences converge.

1. $\left\{\frac{4n-1}{5n+2}\right\}$
2. $\left\{\frac{6-4n^2}{1+n^2}\right\}$
3. $\left\{\frac{3^n}{1+3^n}\right\}$
4. $\{n \cdot 5^{-n}\}$
5. $\{e^{1/n}\}$
6. $\left\{\frac{\ln n}{n^n}\right\}$
7. $\left\{\frac{\ln}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right\}$
8. $\left\{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right\}$
9. $\{\tan^{-1} n\}$
10. $\left\{\frac{\log_e (n+3)}{n+3}\right\}$
11. $(0.8), (0.8)^2, (0.8)^3, \ldots$
12. $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots$
13. The sequence $\{a_n\}$ is defined by

$$a_1 = 2, a_2 = 1, a_{n+1} = \left(1 - \frac{1}{n^2}\right) a_n, n \geq 2$$

Show that it converges.

14. Show that $\left\{ \int_1^n e^{-t^2} dt \right\}$ converges.

[Hint : $e^{x^2} \geq e^x$ for all $x \geq 1$ i.e. $e^{-x^2} \leq e^{-x}$]

## 5.3 INFINITE SERIES

The concept of series is related to the concept of sequence. If $\{a_n\}$ is the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$, then the sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \quad \ldots\ldots (1)$$

is called an infinite series or simply a series. The $a_k$, $k = 1, 2, 3, \ldots$ are called the terms of the series, $a_n$ is called the general term. The above series (1) can be written compactly as

$$\sum_{k=1}^{\infty} a_K, \text{ or briefly for convenience } \Sigma a_K.$$

**EXAMPLE** The decimal representation for the rational number $\frac{1}{9}$ is, infact an infinite series as explained below.

$$\frac{1}{9} = 0.111111 \ldots \infty$$

$$= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\cdots + \frac{1}{10^n} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{10^k}$$

Here the problem we want to discuss is that when does an infinite series "add up" to a number? So this series will give us the number $\frac{1}{9}$.

But the series $\sum_{k=1}^{\infty} 10^k = 10 + 10^2 + 10^3 + \cdots\cdots$ would have no sum.

We can say this in the mathematical language that $\sum_{k=1}^{\infty} 10^k$ does not converge to any number.

The concept of convergence of an infinite series is defined in terms of the convergence of a special kind of sequence.

### 5.3.1 SEQUENCE OF PARTIAL SUMS

The sequence of partial sums can be associated with every infinite series as defined below.

Let $\{S_n\}$ is the sequence of partial sums.

Such that,

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The general term of this sequence of partial sums is,

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n.$$

This is also called the nth partial sum of the series $\sum_{k=1}^{\infty} a_k$.

**EXAMPLE** The sequence of partial sums for $\sum_{k=1}^{\infty} \frac{1}{10^k}$ is,

$$S_1 = \frac{1}{10}$$

$$S_2 = \frac{1}{10} + \frac{1}{10^2}$$

$$S_3 = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}$$

$$S_4 = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4}$$

$$\vdots$$

$$S_n = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n}$$

$$\vdots$$

When n is very large $S_n$ will give a good approximation to $\frac{1}{9}$. So it is reasonable to write

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{1}{10^k} = \sum_{k=1}^{\infty} \left(\frac{1}{10^k}\right) = \frac{1}{9}.$$

## 5.3.2 CONVERGENT SERIES

## 5.3.3 DEFINITION

An infinite series $\sum_{k=1}^{\infty} a_k$ is said to be convergent, if the sequence of partial sums $\{S_n\}$ converges; i.e.

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = S$$

The number S is the sum of the series.

If $\lim_{n \to \infty} S_n$ does not exist, the series is said to be divergent.

**EXAMPLE** Show that the series $\sum_{K=1}^{\infty} \frac{1}{K(K+1)}$ is convergent.

**SOLUTION** The general term of the series

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \text{ by partial fraction.}$$

Thus the nth partial sum,.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$S_n = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{1} - \frac{1}{n+1}$$

$$\lim_{n \to \infty} (S_n) = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right)$$

$$= 1 - 0 = 1, \quad \lim_{n \to \infty} \left(\frac{1}{n+1}\right) = 0$$

$$\therefore \lim_{n \to \infty} S_n = 1, \quad \therefore \text{ series is convergent.}$$

We usually write $\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = 1$

**NOTE** Because of the manner in which the general term of the sequence of partial sums collapses to two terms i.e. $1 - \frac{1}{n+1}$, such series are usually called TELESCOPING SERIES.

e.g. $\sum_{K=1}^{\infty} \frac{1}{(K+4)\ (K+5)}$, $\Sigma \frac{1}{(K+3)(K+4)}$, etc.

### 5.3.4 GEOMETRIC SERIES

A series of the form, $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = a \sum_{k=1}^{\infty} r^{k-1}$

is called a Geometric series.

**5.3.5 THEOREM**

A geometric series $\sum_{k=1}^{\infty} ar^{k-1}$, $a \neq 0$, converges to $\frac{a}{1-r}$ if $|r| < 1$.

**PROOF** The general term $S_n$ of sequence of partial sums of $\sum_{k=1}^{\infty} ar^{k-1}$ is given by

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

Now $\lim_{n \to \infty} r^n \to 0$, if $|r| < 1$.

Consequently $\lim_{n \to \infty} S_n = \frac{a}{1-r}$. So the series converges for $|r| < 1$.

If $|r| > 1$, $\lim_{n \to \infty} r^n$ does not exist.

Consequently $\lim_{n \to \infty} S_n$ does not exist.

So the series diverges for $|r| > 1$.

If $r = \pm 1$

$S_n = a + a + a + \cdots + a = na$, for $r = 1$

$\lim_{n \to \infty} S_n = a \lim_{n \to \infty} (n)$, which does not exist.

Similarly $S_n = a - a + a - a + \cdots + (-1)^{n-1} a$

$= 0$ or $a$, for n even or odd.

$\therefore \lim_{n \to \infty} S_n$ does not exist.

Hence $\sum_{k=1}^{\infty} ar^{k-1}$ diverges for $|r| \geq 1$, converges for $|r| < 1$.

**EXAMPLES** $\sum_{k=1}^{\infty} \left(\frac{-1}{3}\right)^{k-1}$, $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+1}$ converge.

while $\sum_{k=1}^{\infty} 6(3/2)^{k-1}$ diverges.

### 5.3.6 HARMONIC SERIES

$\sum_{k=1}^{\infty} \frac{1}{k}$ is called harmonic series. This is very important example of divergent series. We prove it as given below.

Here, $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

Thus,

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$= S_n + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.$$

Now consider $n + 1 < n + n$, for $n > 1$

$$\therefore \frac{1}{n+1} > \frac{1}{2n} \text{ or } \frac{1}{2n} < \frac{1}{n+1}$$

$$\frac{1}{2n} < \frac{1}{n+2}$$

$$\frac{1}{2n} < \frac{1}{n+3}$$

$$\equiv \equiv \equiv \equiv \equiv$$

$$\therefore S_{2n} \geq S_n + \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$S_{2n} \geq S_n + \frac{n}{2n} = S_n + \frac{1}{2}.$$

This inequality shows that the sequence of partial sums is not bounded.

So the conclusion is that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

### 5.3.7 THEOREM

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \to \infty} a_n = 0$.

By giving a counter example prove that this condition is necessary but not sufficient.

**PROOF** $S_n - S_{n-1} = a_1 + a_2 + a_3 + \dots + a_n$

$$- [a_1 + a_2 + a_3 + \dots + a_{n-1}] = a_n$$

$$\lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} a_n$$

$\{S_n\}$ is convergent. $\therefore \lim_{n \to \infty} S_n = S$ (say)

Also $\lim_{n \to \infty} S_{n-1} = S$

$\therefore$ Lim $(S_n - S_{n-1}) = S - S = 0 = \lim_{n \to \infty} a_n$

Hence $\sum_{k=1}^{\infty} a_k$ converges implies $\lim_{n \to \infty} a_n = 0.$

But the converse is not true.

For example $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges,

though $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$

The nth term test for Divergence.

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

**EXAMPLE 1** Consider the series $\sum_{k=1}^{\infty} \frac{4k + 5}{2 - 7k}$.

Here $a_n = \frac{4n + 5}{2 - 7n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{4n + 5}{2 - 7n}$$

$$= \lim_{n \to \infty} \left[ \frac{4 + \frac{5}{n}}{\frac{2}{n} - 7} \right] = \frac{4}{-7} = \frac{-4}{7} \neq 0.$$

So by divergence test $\sum_{k=1}^{\infty} \frac{4k + 5}{2 - 7k}$ is divergent.

**We state the following without proof.**

1. If C is any constant, then $\sum_{k=1}^{\infty} a_K$ and $\sum_{k=1}^{\infty} ca_K$ both converge or diverge together.

2. If $\sum_{k=1}^{\infty} a_K$ converges to $S_1$ and $\sum_{k=1}^{\infty} b_K$ converges to $S_2$ respectively then $\sum_{k=1}^{\infty} (a_K + b_K)$ converges to $S_1 + S_2$.

3. If $\sum_{k=1}^{\infty} a_K$ converges and $\sum_{k=1}^{\infty} b_K$ diverges,

then $\sum_{k=1}^{\infty} (a_K + b_K)$ diverges.

**EXAMPLE 2** Discuss $\sum_{k=1}^{\infty}\left[\frac{1}{(k+4)(k+3)} + \frac{1}{k}\right]$ for convergence or divergence.

**SOLUTION** Let $a_K = \frac{1}{(k+4)(k+3)}$, $b_K = \frac{1}{k}$

$\therefore$ We want to discuss now

$$\sum_{k=1}^{\infty} a_K + \sum_{k=1}^{\infty} b_K$$

where $\sum_{k=1}^{\infty} a_K = \sum_{k=1}^{\infty} \frac{1}{(k+3)(k+4)}$ is a telescoping series

$$\therefore S_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right)$$

$$= \frac{1}{4} - \frac{1}{n+4}$$

So, $\lim_{n \to \infty} S_n = \frac{1}{4}$, $\lim_{n \to \infty} \frac{1}{n+4} = 0$

$\therefore \sum_{1}^{\infty} a_K$ is convergent.

$\sum_{k=1}^{\infty} b_K = \sum_{k=1}^{\infty} \frac{1}{k}$ is harmonic series,

So divergent.

The conclusion is, that the given series, being the sum of two series one convergent and the other divergent, is divergent.

REMARK The deletion of the first several terms of a series does not affect the convergence or divergence of that series.

## EXERCISE 5.3

In the following series, determine whether they converge or diverge. If convergent, find the sum of the series.

1. $\sum_{k=1}^{\infty} 2\left(\frac{1}{5}\right)^{K-1}$

2. $\sum_{r=1}^{\infty} 5^r 4^{-r}$

3. $\sum_{k=1}^{\infty} (\pi/3)^{K-1}$

4. $\sum_{n=1}^{\infty} 10000\ (0.9)^{n-1}$

5. $\sum_{k=1}^{\infty} \frac{1}{(\sqrt{3} - \sqrt{2})^{K-1}}$

6. $\sum_{k=1}^{\infty} \left(\frac{\sqrt{5}}{1 + \sqrt{5}}\right)^{K-1}$

7. $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

8. $\sum_{k=1}^{\infty} \left[\left(\frac{1}{3}\right)^{K-1} + \left(\frac{1}{4}\right)^{K-1}\right]$

9. $\sum_{n=1}^{\infty} \frac{2^n - 1}{4^n}$

10. $\sum_{n=1}^{\infty} 5$

11. $\sum_{n=1}^{\infty} (-1)^{n+1}$

12. $\sum_{n=1}^{\infty} \log_e\left(\frac{n}{3n + 1}\right)$

13. $\sum_{n=1}^{\infty} \left[\frac{1}{2^n} + \frac{7}{n}\right]$

14. $\sum_{n=1}^{\infty} n \sin\left(\frac{6}{n}\right)$

15. $\sum_{n=3}^{\infty} \log_e n$

16. Determine whether the sum of two divergent series is necessarily divergent.

17. Suppose the sequence $\{a_n\}$ converges to a number $L \neq 0$. Show that $\sum_{k=1}^{\infty} a_K$ diverges.

18. Find all values of $\theta$ in $]-\pi/2, \pi/2[$ for which

$$\lim_{n\to\infty}\left(\frac{1}{1 - \tan\theta} - \sum_{k=0}^{n} \tan^k \theta\right) = 0.$$

**19.** Determine whether the following argument is valid :-

If $S = 1 + 2 + 4 + 8 + 16 + \cdots$,

then $2S = 2 + 4 + 8 + 16 + 32 + \cdots = S - 1$,

$$2S = S - 1 \Rightarrow S = -1.$$

**20.** Discuss whether $\sum_{k=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{1}{k}\right)$ converges or diverges.

## 5.4 SERIES OF POSITIVE TERMS

We have observed that telescoping series or geometric series can easily be discussed to prove their convergence or divergence, directly from the sequence of partial sums. But for the other series we feel difficulty to discuss their convergence or divergence by considering the partial sums sequence $\{S_n\}$. So for this purpose we discuss by means of a TEST that utilizes only the terms of the series. We explain some tests in the coming section for this task.

### 5.4.1 INTEGRAL TEST

Suppose we define a function which has the following three properties.

(i) f is continuous function for $x \geq 1$

(ii) f is non-negative function

(iii) f is decreasing function

such that $f(k) = a_K$ for $k \geq 1$

If f has the above properties then,

(a) $\int_1^{\infty} f(x)\,dx$ convergence $\Rightarrow \sum_{k=1}^{\infty} a_K$ converges.

(b) $\int_1^{\infty} f(x)\,dx$ divergence $\Rightarrow \sum_{k=1}^{\infty} a_K$ diverges.

**PROOF** If the graph of f is given in the figure (B) , then by considering areas of the rectangles shown in the figure, we observe

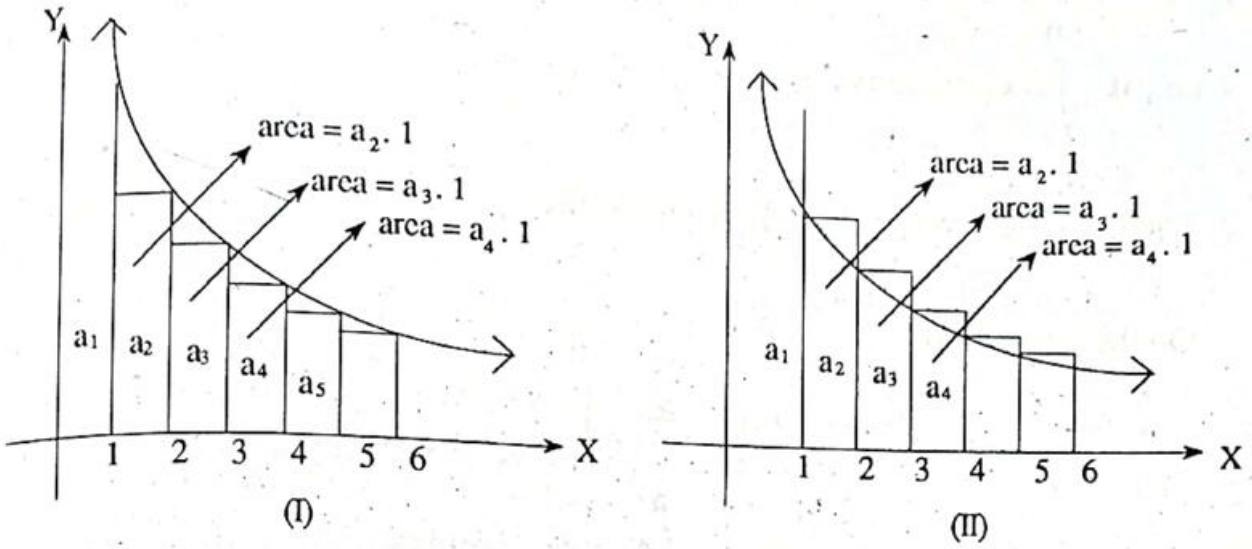



Figure (B)

$$0 \le a_2 \cdot 1 + a_3 \cdot 1 + a_4 \cdot 1 + \cdots + a_n \cdot 1 \le \int_1^n f(x)\, dx \qquad \ldots\ldots (1)$$

This can be observed with reference to (I).

$$\int_1^n f(x)\, dx \le a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots + a_{n-1} \cdot 1 \qquad \ldots\ldots (2)$$

This can be observed with reference to (II).

Let us denote,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

(1) and (2) can be written as

$$0 \le S_n - a_1 \le \int_1^n f(x)\, dx \le S_{n-1} = S_n - a_n$$

From the inequality

$$S_n - a_1 \le \int_1^n f(x)\, dx$$

It is apparent $\lim_{n \to \infty} S_n$ exists

Whenever $\lim_{n \to \infty} \int_1^n f(x)\, dx$ exists.

i.e., if $\int_1^n f(x)\,dx$ converges

Then, $\{S_n\}$ converges i.e. $\sum_{k=1}^{\infty} a_k$ converges.

On the other hand

$$S_{n-1} \geq \int_1^n f(x)\,dx,$$

So the conclusion is if $\lim_{n \to \infty} \int_1^n f(x)\,dx$ diverges.

Then $\lim_{n \to \infty} S_{n-1}$ fails to exist.

i.e. $\sum_{k=1}^{\infty} a_K$ diverges.

**NOTE** In the integral test, if the positive term series is of the form $\sum_{k=m}^{\infty} a_K$, we then use the integral as

$$\int_m^{\infty} f(x)\,dx, \text{ where } f(k) = a_K.$$

**EXAMPLE** Test for convergence $\sum_{n=4}^{\infty} \frac{\log n}{n}$

**SOLUTION** Here the function,

$$f(x) = \frac{\log x}{x}$$

$$f'(x) = \frac{1 - \log x}{x^2} \text{ which is}$$

negative for $x \in [\,4, \infty\,[$

i.e. f(x) is decreasing.

f(x) is continuous and non-negative.

Now $\int_4^{\infty} \frac{\log x}{x}\, dx = \lim_{t \to \infty} \int_4^{t} \frac{\log x}{x}\, dx.$

$= \frac{1}{2}\left[(\log x)^2\right]_4^t$, where $\lim_{t \to \infty} \int_4^{\infty} \frac{\log x}{x}\, dx$

$= \frac{1}{2} \cdot \lim_{t \to \infty} \left[(\log t)^2 - (\log 4)^2\right] = \infty$

This shows that the integral diverges.

So $\sum_{k=4}^{\infty} \frac{\log k}{k}$ also diverges.

### 5.4.2 P-SERIES

The P-series is of the type $\sum_{k=1}^{\infty} \left(\frac{1}{k^p}\right)$

i.e. $\sum_{k=1}^{\infty} \frac{1}{(k)^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

The Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a p-series where $p = 1$.

### 5.4.3 THEOREM

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

**PROOF** Here $f(x) = \frac{1}{x^p}$ for the purpose of integral test where $x \in [1, \infty[$

$f(x)$ is continuous, positive and decreasing, because $f'(x) = \frac{-p}{x^{p+1}} < 0$ for $x \in [1, \infty]$.

$\therefore \int_1^{\infty} f(x)\, dx = \int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^p}\, dx$

$$= \lim_{t \to \infty} \left\{ \frac{1}{1-p} \cdot \left[ \frac{1}{x^{p-1}} \right]_1^t \right\}$$

$$= \frac{1}{1-p} \left[ \frac{1}{t^{p-1}} - 1 \right] \text{ when } t \to \infty$$

if $p > 1$, $p - 1 > 0$

then $\lim_{t \to \infty} \frac{1}{t^{p-1}} = 0$

The integral converges to $\frac{1}{p-1}$.

So the p-series converges for $p > 1$

If $p \leq 1$, the integral diverges.

So the p-series diverges for $p \leq 1$.

Remember the Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges because it is a p-series for $p = 1$.

**EXAMPLE**

(a) The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges since $p = 1/2 < 1$.

(b) The series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges since $p = 4 > 1$.

### 5.4.4 COMPARISON TEST

If we know about the convergence or divergence of a series $\sum_{k=1}^{\infty} b_k$, called usually the **TEST SERIES**, then it is often possible to determine convergence or divergence of a series $\sum_{k=1}^{\infty} b_k$ by comparing its terms with the terms of the **test series.**

### 5.4.5 THEOREM (Comparison Test)

Suppose $\sum_{K=1}^{\infty} a_K$ and $\sum_{K=1}^{\infty} b_K$ are series of positive terms.

(i) If $\sum_{k=1}^{\infty} b_K$ converges and $a_K \leq b_K$ for every positive integer k, then $\sum_{k=1}^{\infty} a_K$ converges.

(ii) If $\sum_{k=1}^{\infty} b_K$ diverges and $a_K \geq b_K$ for every positive integer k, then $\sum_{k=1}^{\infty} a_K$ diverges.

PROOF Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

and $T_n = b_1 + b_2 + b_3 + \cdots + b_n$

be the general terms of the sequence of partial sums for $\Sigma a_K$ and $\Sigma b_K$ respectively.

If $\Sigma b_K$ is convergent series for which $a_K \leq b_K$, then

$$a_1 \leq b_1$$

$$a_2 \leq b_2$$

$$a_3 \leq b_3$$

$$\equiv \quad \equiv \quad \equiv$$

$$a_n \leq b_n$$

$a_1 + a_2 + a_3 + \cdots + a_n \leq b_1 + b_2 + b_3 + \cdots + b_n$, [adding]

or $S_n \leq T_n$.

$\Sigma b_K$ is convergent $\Rightarrow \lim_{n \to \infty} (T_n)$ exists.

$$\lim_{n \to \infty} S_n \leq \lim_{n \to \infty} T_n$$

$\therefore$ $\{S_n\}$ is bounded monotonic increasing sequence.

$\therefore$ $\{S_n\}$ is convergent.

Therefore $\Sigma a_K$ is convergent.

(ii) If $\Sigma b_K$ diverges and $a_K \geq b_K$,

then $S_n \geq T_n$. Since $T_n$ increases without bound, so does $S_n$.

Hence, $\{S_n\}$ is not bounded.

$\therefore$ $\Sigma a_K$ is divergent.

**NOTE** In general, if $\Sigma U_K$ and $\Sigma V_K$ are two series for which $U_K \leq V_K$ for all k, we say $\Sigma U_K$ is **DOMINATED** by $\Sigma V_K$. The comparison test says that if $\Sigma a_n$ is Dominated by a convergent series then it converges. On the other hand if $\Sigma a_K$ dominates a divergent series then it is divergent.

**EXAMPLE 1** Test for convergence $\sum_{k=1}^{\infty} \frac{k}{k^3 + 11}$.

**SOLUTION** We observe

$$\frac{k^3 + 4}{k} \geq \frac{k^3}{k} = k^2$$

$$\text{or } \frac{k}{k^3 + 4} \leq \frac{1}{k^2}$$

So we consider the test series $\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \Sigma \frac{1}{k^2}$ is convergent being the p-series for which $p = 2 > 1$ and the given series

$$\Sigma \frac{k}{k^3 + 4} \leq \Sigma \frac{k}{k^2}$$

i.e. It is dominated by a convergent series, $\therefore$ By comparison test (L.T.), it is convergent.

**EXAMPLE 2** Test for convergence $\sum_{k=1}^{\infty} \frac{\log_e (k+2)}{k}$.

**SOLUTION** Since $\log_e (k + 2) > 1$ for $k \geq 1$,

$$\therefore \frac{\log_e (k+2)}{k} > \frac{1}{k}, \text{ [Dividing } bk \geq 1]$$

$$\Sigma \left[\frac{\log_e (k+2)}{k}\right] \geq \Sigma \frac{1}{k}$$

but $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series.

$\therefore$ The given series dominates a divergent series.

Hence by L.T. $\sum_{k=1}^{\infty} \frac{\log_e (k+2)}{k}$ is divergent.

Another kind of comparison test is called limit comparison test stated below.

5.4.6 **THEOREM (Limit Comparison Test)**

Let $\sum_{k=1}^{\infty} a_K$ and $\sum_{k=1}^{\infty} b_K$ are positive terms series and that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \quad \text{(say)}$$

(i) If L is +ve constant, then the two series converge or diverge together.

(ii) If $L = 0$ and $\sum b_K$ converge, then $\sum a_K$ converges.

(iii) If $L = \infty$ and $\sum b_K$ diverges, then $\sum a_K$ diverges.

PROOF

(i) $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = L > 0$, (given)

we can choose $n \geq N$ for some positive integer N, such that

$$\left| \frac{a_n}{b_n} - L \right| < \epsilon \text{ , for } n \geq N,$$

The positive integer N depends upon $\epsilon > 0$.

or $-\epsilon < \frac{a_n}{b_n} - L < \epsilon$ , for $n \geq N$.

$$L - \epsilon < \frac{a_n}{b_n} < L + \epsilon \qquad \ldots\ldots (A)$$

This inequality results in

$$a_n < (L + \epsilon)\, b_n, \quad \forall\, n \geq N.$$

$$\therefore \sum_{k=N}^{\infty} a_K < (L + \epsilon) \sum_{k=1}^{\infty} b_K$$

If $\sum b_K$ is convergent.

Then $\sum a_K$ is dominated by a convergent series. Hence $\sum a_K$ is convergent by comparison test.

If we consider

$$(L - \epsilon) < \frac{a_n}{b_n}, \quad \forall\, n \geq N$$

$$(L - \epsilon)\, b_n < a_n$$

$$(L-\epsilon)\sum_{k=N}^{\infty} b_K < \sum_{k=N}^{\infty} b_K$$

If $\sum b_K$ is divergent,

Then $(L-\epsilon)\sum_{k=N}^{\infty} b_K$ is divergent.

Consequently $\sum a_n$ is dominating a divergent series. Hence $\sum a_n$ is divergent by comparison test.

(ii) If $L = 0$

The above inequality ... (A) becomes,

$$-\epsilon < \frac{a_n}{b_n} < \epsilon \ , \forall\ n \geq N$$

$a_n < \epsilon\ b_n \ , \quad \forall\ n \geq N.$

$$\therefore \quad \sum_{k=N}^{\infty} a_K < \epsilon \sum_{k=N}^{\infty} b_K$$

If $\sum b_K$ is convergent,

then by the previous argument, that $\sum a_K$ is dominated by a convergent series, implies $\sum a_K$ is convergent.

(iii) If $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \infty$

i.e. if G is very large number

Then $\frac{a_n}{b_n} > G$, for $n \geq N$.

Where N depends on G.

$$a_n > G\, b_n$$

$$\sum_{n=N}^{\infty} a_n > G \sum_{n=N}^{\infty} b_n \qquad \ldots\ldots (B)$$

It is given $\sum b_K$ diverges.

So is $G\ \Sigma b_K$, i.e. $G\ \Sigma b_K$ also diverges.

Consequently $\sum_{k=1}^{\infty} a_K$ is dominating a divergent series.

Hence $\Sigma a_K$ is divergent, by comparison test.

**NOTE** The limit comparison test or briefly L.C.T. is often applicable to series for which the comparison test is inconvenient.

**EXAMPLE 1** Test the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^3 - 5k^2 + 1}$

**SOLUTION** It is very difficult to locate the testing series $\Sigma b_n$, so it is difficult to apply the C.T.

But we observe $\frac{1}{n^3 - 5n^2 + 1}$ behaves like $\frac{1}{n^3}$ when n is very large.

However, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent because here $p = 3 > 1$.

So, $a_n = \frac{1}{n^3 - 5n^2 + 1}$ and $b_n = \frac{1}{n^3}$

and we have $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{1}{n^3 - 5n^2 + 1} \div \frac{1}{n^3}\right)$

$$= \lim_{n \to \infty} \left(\frac{n^3}{n^3 - 5n^2 + 1}\right) = \lim_{n \to \infty} \left(\frac{1}{1 - \frac{5}{n} + \frac{1}{n^3}}\right)$$

$$= 1 > 0$$

So, $\Sigma a_K$ and $\Sigma b_K$ behaves alike.

But $\sum_{k=1}^{\infty} \frac{1}{k^3} = \Sigma b_K$ is convergent.

Hence by L.C.T., the given series is convergent.

**NOTE** In such question $\sum_{k=1}^{\infty} b_K$, the testing series is obtained by observing the "degree behavior" of $\sum_{k=1}^{\infty} b_K$, for large values of k. In brief to find $b_n$ we need only examine the quotient of the highest powers of n in the numerator and denominator of $a_n$.

**EXAMPLE 2** Test for convergence $\sum_{k=1}^{\infty} \frac{k}{\sqrt[3]{9k^5 + 6}}$

**SOLUTION** For large values of n,

$$a_n = \frac{n}{\sqrt[3]{9n^5 + 6}} \text{ behaves like a constant multiple of}$$

$$\frac{n}{n^{5/3}} = \frac{1}{n^{2/3}}$$

Thus we consider the testing series

$$\sum_{k=1}^{\infty} b_K = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}} \quad , \quad \text{which is a p-series and divergent because}$$

$$p = \frac{2}{3} < 1.$$

As a testing series

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{n}{\sqrt[3]{9n^5 + 6}}}{\frac{1}{n^{2/3}}}$$

$$= \lim_{n \to \infty} \frac{n^{5/3}}{(9n^5 + 6)^{1/3}}$$

$$= \lim_{n \to \infty} \frac{1}{\left(9 + \frac{6}{n^5}\right)^{1/3}} = \frac{1}{\sqrt[3]{9}} \neq 0$$

Thus by L.C.T. $\Sigma a_n$ is convergent.

## 5.4.7 REMARKS

1. Remember the value of convergent improper integral $\int_1^{\infty} f(x)dx$ is not equal to the sum of the infinite series.
2. The result of the integral test holds if f(x) does not begin to decrease until $x \geq N$. For example, $\sum_1^{\infty} \left(\frac{\log_e k}{k}\right)$ defines the function $f(x) = \frac{\log x}{x}$.

Here $f'(x) = \frac{1 - \log_e x}{x^2} < 0$, for $x \geq 3$.

i.e. f(x) decreases in the interval $[3, \infty[$.

But we can use $\int_1^{\infty} \frac{\log_e x}{x}\, dx$ for integral test though $\left(\frac{\log_e x}{x}\right)$ is only decreasing function for $x \geq 3$.

3. The deletion or omission of few first terms of the series does not effect the convergence or divergence of an infinite series.

4. In the application of Basic Comparison test the following table is helpful.

| Comparison of Terms | Test Series $\Sigma b_K$ | Conclusion about $\Sigma a_K$ |
|---|---|---|
| $a_K \leq b_K$ | Converges | Converges |
| $a_K \leq b_K$ | diverges | No conclusion |
| $a_K \geq b_K$ | diverges | diverges |
| $a_K \geq b_K$ | converges | No conclusion |

## EXERCISE 5.4

Use integral test, comparison test or limit comparison test to discuss whether the following series are convergent or divergent.

1. $\sum_{k=1}^{\infty} \frac{1}{k^{1.2}}$
2. $\sum_{k=1}^{\infty} \frac{1}{k^{0.77}}$
3. $\sum_{k=1}^{\infty} \frac{1}{2k-7}$
4. $\sum_{k=1}^{\infty} \frac{1}{10 + \sqrt{k}}$
5. $\sum_{k=1}^{\infty} \frac{k}{2k+1}$
6. $\sum_{k=1}^{\infty} \frac{1}{k^2 + 7}$
7. $\sum_{k=1}^{\infty} ke^{-k^2}$
8. $\sum_{k=1}^{\infty} \frac{\tan^{-1}k}{1+k^2}$
9. $\sum_{k=1}^{\infty} \frac{1}{(k+6)(k+7)}$
10. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+7k+8}$
11. $\sum_{k=1}^{\infty} \frac{\log k}{k^2+1}$
12. $\sum_{k=1}^{\infty} \frac{\log k}{k^{2+1}}$
13. $\sum_{k=2}^{\infty} \frac{1}{k(\log k)^2}$
14. $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\log_e n}}$
15. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3n+2}}$

16. $\sum_{n=1}^{\infty} \frac{n^2-n+2}{3n^5+n^2+7}$

17. $\sum_{k=1}^{\infty} \frac{2+\sin k}{\sqrt[3]{k^4+1}}$

18. $\sum_{k=1}^{\infty} \frac{3}{2+\sin k}$

19. $\sum_{k=1}^{\infty} \frac{1+8^K}{3+10^K}$

20. $\sum_{k=1}^{\infty} \frac{(1.2)^K}{5^K}$

21. $\sum_{k=1}^{\infty} \frac{1}{3^K+\sqrt{k}}$

22. $\sum_{k=1}^{\infty} \frac{1+5^K}{3^K}$

23. $\sum_{k=2}^{\infty} \frac{1}{\log k}$

24. $\sum_{k=1}^{\infty} \frac{k+e^{-K}}{5^K(k+9)}$

25. $\sum_{j=1}^{\infty} \frac{j^{-1}+1}{10^j}$

26. $\sum_{k=1}^{\infty} \log_e\left(3+\frac{1}{3^K}\right)$

27. $\sum_{k=2}^{\infty} \frac{k+\log k}{k^3+2k-1}$

28. $\sum_{k=1}^{\infty} \frac{\sin(1/k)}{k}$

29. $\sum_{k=1}^{\infty} \frac{\sqrt{k+2}}{(k^9+7)^{1/3}}$

30. $\sum_{k=1}^{\infty} \frac{5k^2+k-k^{-1}}{2k^3+2k^2+8}$

31. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

32. $\sum_{k=1}^{\infty} \tan\left(\frac{1}{k}\right)$

33. $\frac{1}{2\cdot3}+\frac{2}{3\cdot4}+\frac{3}{4\cdot5}+\frac{4}{5\cdot6}+\cdots\cdots$

34. $\frac{1}{1\cdot3}+\frac{1}{2\cdot9}+\frac{1}{3\cdot27}+\frac{1}{4\cdot81}+\cdots\cdots$

35. $2+\frac{3}{5}+\frac{4}{10}+\frac{5}{17}+\cdots\cdots$

## 5.5 RATIO TEST AND ROOT TEST

Ratio test is very helpful when $a_K$ involves factorials, Kth power of a constant, and sometimes, Kth power of K.

### 5.5.1 THEOREM (Ratio-Test)

Suppose $\sum_{k=1}^{\infty} a_K$ is a positive terms series such that $\lim_{n \to \infty} \left[\frac{a_{n+1}}{a_n}\right] = L$

(1) If $L < 1$, $\sum_{k=1}^{\infty} a_K$ converges.

(2) If $L > 1$, $\sum_{k=1}^{\infty} a_K$ diverges.

(3) If $L = 1$, the test fails, i.e. no conclusion.

**PROOF** (i) Let $r > 0$ such that

$$0 \le L \le r < 1$$

For sufficient large n, say $n \geq N$,

where N is a +ve integer,

$$\frac{a_{n+1}}{a_n} < r,$$

i.e.

$$a_{n+1} < r\,a_n \quad , \quad n \geq N.$$

This inequality implies

$$a_{N+1} < r\,a_N$$

$$a_{N+2} < r\,a_{N+1} < a_N r^2$$

$$a_{N+3} < r\,a_{N+2} < a_N r^3$$

and so on.

Thus

$$a_{N+1} + a_{N+2} + a_{N+3} + \cdots\cdots < a_N [r + r^2 + r^3 + \cdots]$$

$$\text{or} \qquad \sum_{k=N+1}^{\infty} a_K < \sum_{k=1}^{\infty} a_N r^K$$

Now in $\sum_{k=N+1}^{\infty} a_k$ we have deleted N finite numbers of terms from $\sum_{k=1}^{\infty} a_k$.

$$\sum_{k=1}^{\infty} a_N r^K = a_N \sum_{r=1}^{\infty} r^K$$

which is geometric series with $|r| < 1$ hence convergent.

$\sum_{k=N+1}^{\infty} a_k$ is dominated by convergent series. Hence convergent.

So the original series is also convergent.

(ii) Here $L > 1$.

Let r be such that

$$1 < r < L.$$

Then for sufficiently large values of n, say $n \geq N$, where N is a +ve integer,

$$\frac{a_{n+1}}{a_n} > r$$

or $a_{n+1} > r a_n$ , for $n \geq N$.

$a_{N+1} > r a_N$

$a_{N+2} > r a_{N+1} > r^2 a_N$

$a_{N+3} > r a_{N+2} > r^3 a_N$

and so on.

≡ ≡ ≡ ≡ ≡ ≡ ≡

$$\therefore \quad \sum_{k=N+1}^{\infty} a_K > a_N \sum_{k=1}^{\infty} r^K$$

where $\sum_{k=1}^{\infty} r^K$ is divergent,

because $|r| > 1$.

$\therefore$ $\sum_{k=N+1}^{\infty} a_K$ is dominating a divergent series, Hence by comparison test it is divergent.

Consequently the original series $\sum_{k=1}^{\infty} a_K$ is divergent.

(iii) If $L = 1$, we must apply some other test for conclusion.

**EXAMPLE 1** Test for convergence $\sum_{n=1}^{\infty} \frac{2^n}{\lfloor n}$

**SOLUTION** Here $a_{n+1} = \frac{2^{n+1}}{\lfloor n+1}$

$a_n = \frac{2^n}{\lfloor n}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{\lfloor n+1} \times \frac{\lfloor n}{2^n}$$

$$= \frac{2}{n+1}$$

$$\therefore \quad \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n \to \infty} \left(\frac{2}{n+1}\right) = 0$$

Since $L = 0 < 1$.

So conclusion is that $\sum_{n=1}^{\infty} \frac{2^n}{\lfloor n}$ converges by ratio-test.

**NOTE** Ratio-Test is also called D'Alembert Ratio-Test.

**EXAMPLE 2** Test for convergence $\sum_{n=1}^{\infty} \frac{n^n}{\lfloor n}$

**SOLUTION** Here $a_n = \frac{n^n}{\lfloor n}$

$$a_{n+1} = \frac{(n+1)^{n+1}}{\lfloor n+1}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{\lfloor n+1} \times \frac{\lfloor n}{n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\therefore \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n}\right)$$

$$= e > 1$$

So by ratio test series is divergent.

## 5.5.2 ROOT-TEST

The test is also called Cauchy's Root-Test. If the terms of the series $\sum a_K$ involve only Kth powers, then this test is helpful.

## 5.5.3 THEOREM (Root-Test)

Let $\sum_{k=1}^{\infty} a_K$ is a series of positive terms such that $\lim_{n \to \infty} (a_n)^{1/n} = L$

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$, the series is divergent.

(iii) If $L = 1$, the test fails, and no conclusion.

**PROOF**

(i) $L < 1$.

$$\lim_{n \to \infty} a_n^{1/n} = L \Rightarrow |a_n^{1/n} - L| < \epsilon, \text{ for } n \geq N$$

where $\epsilon > 0$ and N is a +ve integer depending on $\epsilon$.

or $L - \epsilon < a_n^{1/n} < L + \epsilon$, for $n \geq N$

Let $L + \epsilon < r < 1$

$\therefore a_n^{1/n} < r$

or $a_n < r^n$, for $n \geq N$

or $\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} r^n$

but $\sum_{n=N}^{\infty} r^n$ is convergent, being geometric series whose $|r| < 1$.

$\therefore \sum_{n=N}^{\infty} a_n$ is convergent, being dominated by a convergent series.

Consequently the given series $\sum_{k=1}^{\infty} a_k$ is convergent by comparison test.

(ii) If $L > 1$

We consider

$$L - \epsilon < a_n^{1/n}, \text{ for } n \geq N.$$

Let r is a number such that

$$L - \epsilon > r > 1.$$

Then

$$\left.\begin{aligned} & r < L - \epsilon < a_n^{1/n} \\ \text{or} \quad & r < a_n^{1/n} \\ & r^n < a_n \\ & \sum_{n=N}^{\infty} r^n < \sum_{n=N}^{\infty} a_n \end{aligned}\right\} \text{For } n \geq N$$

but $\sum_{-}^{\infty} r^n$ is divergent, being geometric series having common ratio $r > 1$.

$\therefore$ $\sum_{n=N}^{\infty} a_n$ is dominating a divergent series. Hence divergent.

Consequently the given series is divergent.

(iii) If $L = 1$, test fails. No conclusion so we must apply some other test for conclusion.

**EXAMPLE 1** Discuss the convergence of $\sum_{k=1}^{\infty} (\log_e 3)^k$

**SOLUTION** Here $a_n = (\log_e 3)^n$

$$(a_n)^{1/n} = \log_e 3$$

$$\lim_{n \to \infty} (a_n)^{1/n} = \log_e 3 = L \text{ (say).}$$

here $L = \log_e 3 > 1$

$\therefore$ By Root Test the given series is divergent.

**EXAMPLE 2** Test the convergence of

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{\sqrt{k}}\right)^{-k^{3/2}}$$

**SOLUTION**

Here $$a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$= \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$\therefore$ $$(a_n)^{1/n} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n^{1/2}}}$$

$$= \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\text{Now } \lim_{n \to \infty} (a_n)^{1/n} = \frac{1}{\lim_{n \to \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1$$

Therefore, the series converges by Root-Test.

## EXERCISE 5.5

Apply the Ratio-Test or Root-Test to discuss the convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{1}{\lfloor n}$

2. $\sum_{n=1}^{\infty} \frac{2^n}{\lfloor n}$

3. $\sum_{n=1}^{\infty} \frac{\lfloor n}{10^{3n}}$

4. $\sum_{l=1}^{\infty} l\left(\frac{2}{3}\right)^l$

5. $\sum_{l=1}^{\infty} \frac{l^{10}}{(1.1)^l}$

6. $\sum_{k=1}^{\infty} \frac{1}{k^5(0.98)^k}$

7. $\sum_{p=1}^{\infty} \frac{1}{p^p}$

8. $\sum_{k=1}^{\infty} \left(\frac{ke}{k+1}\right)^k$

9. $\sum_{k=1}^{\infty} \frac{4^{k-1}}{k\, 3^{k+2}}$

10. $\sum_{k=1}^{\infty} \frac{k^3\, 2^{k+3}}{7^{k-2}}$

11. $\sum_{n=1}^{\infty} \frac{\lfloor n}{\lfloor 2n}$

12. $\sum_{n=1}^{\infty} \frac{\lfloor 2n}{\lfloor n\ (2n)^n}$

13. $\sum_{k=1}^{\infty} \frac{5^{2k+1}}{k^k}$

14. $\sum_{k=2}^{\infty} \frac{1}{(\log_e k)^k}$

15. $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

16. $\sum_{k=1}^{\infty} \left(1 - \frac{2}{k}\right)^{k^2}$

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{\lfloor n}$

18. $\sum_{n=1}^{\infty} \frac{\lfloor n}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)}$

19. $\sum_{k=1}^{\infty} \frac{\lfloor k\ 3^k}{k^k}$

20. Find the non-negative values of p for which the following series converges.

(a) $\sum_{k=1}^{\infty} kp^k$

(b) $\sum_{k=1}^{\infty} k^2\left(\frac{2}{p}\right)^k$

21. Find all real values of p for which the given series converges

(a) $\sum_{k=1}^{\infty} \frac{k^p}{\lfloor k}$ (b) $\sum_{k=1}^{\infty} \frac{\log_e k}{k^p}$

22. The sequence $\{F_n\}$, 1, 1, 2, 3, 5, 8, 13, 21, 34, ..... is defined by the recursion formula,

$$F_{n+1} = F_n + F_{n-1}, \text{ where } F_1 = 1, F_2 = 1.$$

verify that the general term of the sequence is

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

by showing that this result satisfies the recursion formula.

23. For $F_n$ defined in question No. 22,

prove that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

24. Prove that the series $\sum_{n=1}^{\infty} \{F_n\}^{-1} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \ldots$

converges.

25. Test for convergence the following series

(i) $\sum_{n=1}^{\infty} e^{n-n^2}$ (ii) $\sum_{n=1}^{\infty} \frac{n}{\pi}\left(\frac{\pi}{n}\right)^n$

(iii) $\sum_{n=1}^{\infty} \left[\frac{\lfloor n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]$ (iv) $\sum_{n=1}^{\infty} \left[\frac{2^n \lfloor n}{5 \cdot 8 \cdot 11 \cdot 14 \cdots (3n+2)}\right]$

(v) $\sum_{1}^{\infty} \frac{(a+1)(2a+1)(3a+1)\ldots(na+1)}{(b+1)(2b+1)(3b+1)\ldots(nb+1)}$ , $a > 0, b > 0$

26. Show that the series $\sum_{n=1}^{\infty} \frac{(\lfloor n)^2}{\lfloor 2n} x^n$ converges for $0 < x < 4$.

27. If $x > 0$, find values of x for which the following series converge

(i) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)} x^n$ (ii) $\sum_{n=1}^{\infty} \frac{x^n}{\lfloor n}$

(iii) $\sum_{n=1}^{\infty} \left\{\frac{x^{2n}}{2n}\right\}$

## 5.6 ABSOLUTE AND CONDITIONAL CONVERGENCE

### ALTERNATING SERIES

### 5.6.1 DEFINITION (ABSOLUTE CONVERGENCE)

The series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

**EXAMPLE 1** The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Converges absolutely because $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k^2}\right|$ converges.

**EXAMPLE 2** The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

does not converge absolutely because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. The importance of absolute convergence is given in the theorem below.

### 5.6.2 THEOREM

If $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ also converges.

That is:

**Absolute convergence implies convergence**

**REMARK** The converse of this theorem is false, that is there are series that are convergent but not absolutely convergent.

**PROOF** Since $a_k \le |a_k|$, we have

$$0 \le a_k + |a_k| \le 2|a_k|$$

Since $\sum_{k=1}^{\infty} |a_k|$ converges, we see that $\sum_{k=1}^{\infty} (a_k + |a_k|)$ converges by the comparison test. Then since $a_k = (a_k + |a_k|) - |a_k|$, $\sum_{k=1}^{\infty} a_k$ converges because it is the sum of two convergent series.

**EXAMPLE 3** The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ considered in Example 1 converges since it converges absolutely.

### 5.6.3 ALTERNATING SERIES

A series in which successive terms have opposite signs is called an alternating series.

**EXAMPLE 4** The series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is an alternating series.

**EXAMPLE 5** The series $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \cdots$ is not an alternating series because two successive terms have the same sign.

Let us consider the series of Example 4

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Calculating successive partial sums, we find that

$$S_1 = 1,\ S_2 = \frac{1}{2},\ S_3 = \frac{5}{6},\ S_4 = \frac{7}{12},\ S_5 = \frac{47}{60},$$

it is clear that this series is not diverging to infinity $\left(\text{indeed},\ \frac{1}{2} \le S_n < 1\right)$ and that the partial sums are getting "narrowed down". At this point it is reasonable to suspect that the series converges. But it does not converge absolutely (since the series of absolute values is the harmonic series) and we cannot use any of the tests of the previous section since the terms are not non-negative. The result we need is given in the theorem below.

### 5.6.4 THEOREM

### ALTERNATING SERIES TEST

Let $\{a_k\}$ be a decreasing sequence of positive numbers such that $\lim_{k\to\infty} a_k = 0$

then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1}\ a_k = a_1 - a_2 + a_3 - \cdots$ converges.

**PROOF** Looking at the odd-numbered partial sums of this series. We find that

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_n) + (a_5 - a_6) + \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1}$$

Since $\{a_K\}$ is decreasing all the terms in the parentheses are non-negative, so that $S_{2n+1} \geq 0$ for every n. Moreover,

$$S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3} = S_{2n+1} - (a_{2n+2} - a_{2n+3}),$$

and since $a_{2n+2} - a_{2n+3} \geq 0$, we have $S_{2n+3} \leq S_{2n+1}$

Hence the sequence of odd-numbered partial sums is bounded below by 0 and is decreasing and is convergent. Thus $S_{2n+1}$ converges to some limit L. Consider the sequence of even-numbered partial sums. We find that $S_{2n+2} = S_{2n+1} - a_{2n+2}$ and since $a_{2n+2} \to 0$

$$\lim_{n\to\infty} S_{2n+2} = \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} a_{2n+2} = L - 0 = L,$$

so that the even partial sums also converge to L. Since both the odd and even sums. Converge to L, we see that the partial sums converge to L and the proof is complete.

**EXAMPLE 6** The following alternating series are covnergent by the alternating series test.

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

(b) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$

(c) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5}$

(d) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \frac{1}{2^6}$

### 5.6.5 DEFINITION (CONDITIONAL CONVERGENCE)

An alternating series is called conditionally convergent. If it is convergent but not absolutely convergent.

In above Example No. 6, (a), (b), (c) series are conditionally convergent, but in 6(d) series is absolutely convergent.

## EXERCISE 5.6

Discuss for conditionally convergent, absolutely convergent, divergent etc. The given following series.

1. $\sum_{k=1}^{\infty}(-1)^k$

2. $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\log_e k}$

3. $\sum_{k=2}^{\infty}\frac{(-1)^k k}{\log_e k}$

4. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{5k-4}$

5. $\sum_{k=0}^{\infty}\cos\frac{k\pi}{2}$

6. $\sum_{k=1}^{\infty}\frac{|k}{(-3)^k}$

7. $\sum_{k=1}^{\infty}\frac{k^2}{(-2)^k}$

8. $\sum_{k=2}^{\infty}\frac{(-1)^k k^2}{k^3+1}$

9. $\sum_{k=3}^{\infty}\frac{\sin(k\pi/7)}{k^3}$

10. $\sum_{k=2}^{\infty}\frac{(-1)^k k(k+1)}{(k+2)^3}$

11. $\sum_{k=1}^{\infty}\frac{(-1)^k 2^k}{k}$

12. $\sum_{k=1}^{\infty}\frac{(-1)^k k^k}{kK}$

13. $\sum_{k=2}^{\infty}\frac{(-1)^k(k^2+3)}{k^3+4}$

14. $\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{4+k^2}$

15. $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\sqrt{\log k}}$

6

# FOURIER SERIES

## 6.1 PERIODIC FUNCTIONS

A function f(x) is said to be periodic if there exists a number p > 0 such that

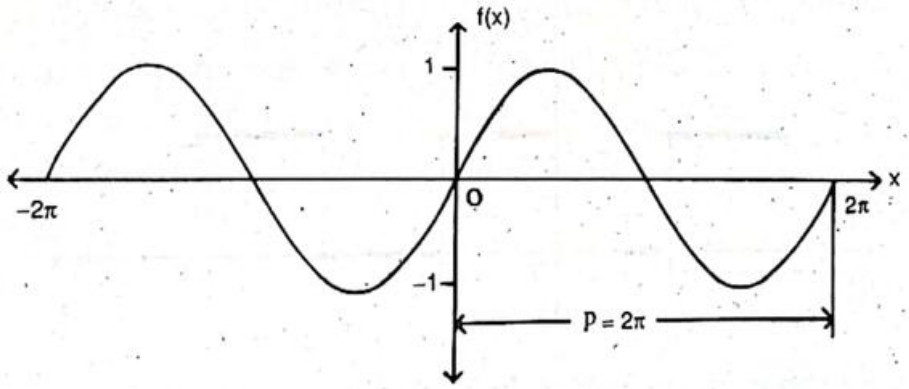
$$f(x + p) = f(x), \qquad \ldots\ldots \text{(i)}$$

for all x belonging to the domain of definition of the function. The least value of p>0 is called the **period** (sometimes called primitive period) of the function f(x).

For example, the function sin x possesses the property

$$\sin x = \sin (x + 2\pi) = \sin (x + 4\pi) = \sin (x + 6\pi) = \ldots\ldots$$

However, $2\pi$ is the least of the periods $2\pi$, $4\pi$, $6\pi$ etc. Hence sin x is a periodic function with period $2\pi$.




Similarly, the period of cos x is $2\pi$ and that of tan x and cot x is $\pi$.

Moreover, the period of sin nx or cos nx, where n is a positive integer, is $\frac{2\pi}{n}$. That is, the period of sin 2x or cos 2x it $\pi$.

If $f_1(x)$ and $f_2(x)$ have the same period p, then the function

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

also has the period p.

But the period of the sum $c_1 \sin x + c_2 \sin 2x$ is $2\pi$ which is, in fact, the L.C.M. of $2\pi$ and $\pi$, the periods of $\sin x$ and $\sin 2x$ respectively.

A constant function $f = c$ is also periodic and has any positive number as its period.

### 6.1.1 PERIODIC EXTENSION OF A FUNCTION

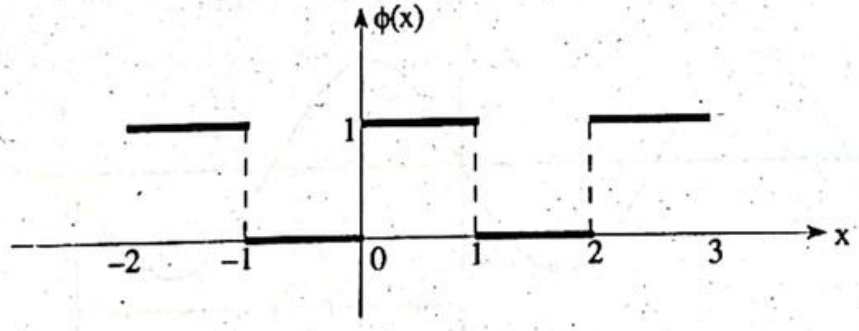
Let a function f(x) be defined in the interval [–p, p] where p is a positive real number. Then the periodic extension $\phi(x)$ of f(x) is defined as

(i) $\phi(x) = f(x)$ , if $x \in [-p, p]$

(ii) $\phi(x + 2p) = \phi(x)$ , for all x.

The condition (i) requires $\phi(x)$ and f(x) to agree on the fundamental interval [–p, p]. The condition (ii) extends the definition of f(x) so that $\phi(x)$ is defined everywhere and is periodic with period 2p.

This means that in order to sketch the periodic extension $\phi(x)$ of a function f(x) we first sketch f(x) for the fundamental interval [–p, p] and then go on repeating the pattern of the graph with period 2p.




The above figure shows the sketch of the periodic extension of

$$f(x) = 0 \quad \text{for } x < 0$$

$$= 1 \quad \text{for } x \geq 0,$$

the fundamental interval being (–1, 1).

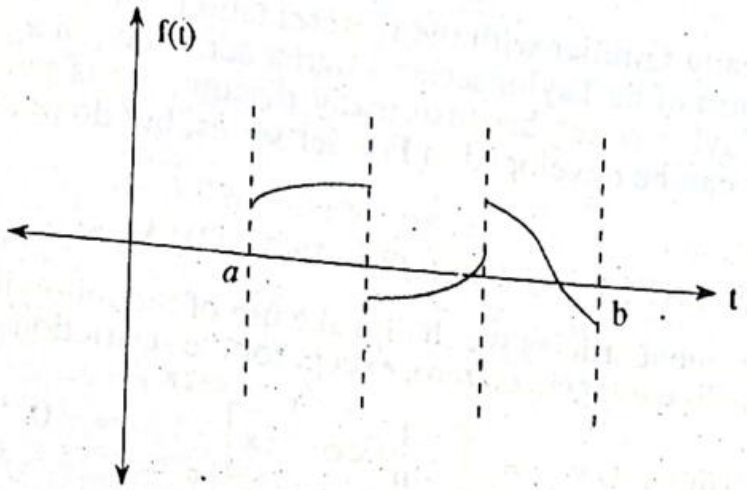
### 6.1.2 PIECEWISE CONTINUOUS FUNCTION

A function f(t) is called piecewise continuous on a finite interval $a \leq t \leq b$ if

(i) it is defined on that interval

(ii) the interval can be divided into a finite number of subintervals in each of which f(t) is continuous

(iii) the limits of f(t) as t approaches the end-points of each subinterval are finite but are not equal.




The figure shows the graph of a piecewise continuous function.

It is obvious that finite jumps are the only discontinuities which a piecewise continuous function may have at the points of sub-division of $[a, b]$.

**NOTE**

(i) A continuous function is piecewise continuous.

(ii) If f(x) is piecewise continuous on $[a, b]$ with the partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

then $\int_a^b f(x)\,dx$ exists and

$$\int_a^b f(x)\,dx = \int_{a_0}^{a_1} f_1(x)\,dx + \int_{a_1}^{a_2} f_2(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f_n(x)\,dx.$$

where on each sub-interval $[a_{i-1}, a_i]$ there is a continuous function $f_i$ such that

$$f(x) = f_i(x) \text{ for } a_{i-1} < x < a_i.$$

## 6.1.3 INTRODUCTION TO FOURIER SERIES

Periodic functions are of common occurrence in various physical phenomena and engineering problems. For example, as in conduction of heat, mechanical vibrations, electric circuits, electromagnetic waves etc. While solving such problems the desired results are often achieved by representing these functions by the sum of an infinite trigonometric series of sines and cosines which are periodic functions with period $2\pi$. Many partial differential equations arising from Physics, Applied mathematics and Engineering can be conveniently solved by means of such a series known as Fourier series after the French mathematician and physicist J. Fourier (1768–1830) who was the first to study such representations in a

systematic manner. The study of conditions under which such expansions are possible and then determination of the coefficients of the series is known as Fourier analysis.

We are already familiar with the representation of an infinitely differentable function in the form of its Taylor series. Fourier series are, in a certain sense, more universal than Taylor series because many discontinuous periodic functions of practical interest can be developed in Fourier series, but do not have Taylor series representations.

## 6.1.4 SOME USEFUL DEFINITE INTEGRALS

In the subsequent articles we shall make use of the following results, where m and n are any positive integers or zero, except for the restrictions indicated.

(i) $\int_c^{c+2\pi} \sin nx\,dx = \left[-\frac{1}{n}\cos nx\right]_c^{c+2\pi} = 0 \quad , \quad n \neq 0$

(ii) $\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \quad , \quad n \neq 0$

(iii) $\int_c^{c+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\sin (m+n)x + \sin (m-n)x]dx$

$= 0 \quad , \quad m \neq n$

(iv) $\int_c^{c+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m-n)x - \cos (m+n)x]dx$

$= 0 \quad , \quad m \neq n$

(v) $\int_c^{c+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m+n)x + \cos(m-n)x]dx$

$= 0 \quad , \quad m \neq n$

(vi) $\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = 0 \quad , \quad n \neq 0$

(vii) $\int_c^{c+2\pi} \sin^2 nx\,dx = \frac{1}{2}\int_c^{c+2\pi} (1 - \cos 2nx)dx = \pi \quad , \quad n \neq 0$

(viii) $\int_c^{c+2\pi} \cos^2 nx\,dx = \frac{1}{2}\int_c^{c+2\pi} (1 + \cos 2nx)dx = \pi \quad , \quad n \neq 0$

(ix) $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$

where $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on.

and $v_1 = \int v\,dx$, $v_2 = \int v_1\,dx$ and so on.

### 6.1.5 DEFINITION

A trigonometric infinite series of the form

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \ldots\ldots \text{(i)}$$

corresponding to a function f(x), is called its **Fourier series** and the constants $a_n$ and $b_n$ are known as Fourier constants or coefficients.

In (i) we have taken the constant term as $\frac{a_0}{2}$ instead of $a_0$ because the Euler's formula for $a_n$, n = 1, 2, 3, ... which we are going to establish soon, will also hold for n = 0.

Since the functions sin nx and cos nx of a Fourier series are periodic with the period $2\pi$, therefore the sum function f(x) of a convergent Fourier series of the form (i) must also be periodic with period $2\pi$. However, it is not necessary that f(x) should be a trigonometric function. A function, with $2\pi$ as its period, will arise if it is arbitrarily defined in an interval of length $2\pi$ and then periodically extended beyond this interval to the left and to the right so as to satisfy $f(x \pm 2\pi) = f(x)$.

### 6.1.6 THE COMPUTATION OF THE FOURIER COEFFICIENTS

**THEOREM** If the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is assumed integrable term-by-term in the interval $c \le x \le c + 2\pi$ and has the sum f(x), then the coefficients $a_n$ and $b_n$ are given by

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \ldots$$

and $$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx, \quad n = 1, 2, 3, \ldots$$

**PROOF** Let f(x) be a periodic function with period $2\pi$ and defined in the interval $c \le x \le c + 2\pi$. Let f(x) be represented in this interval by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots\ldots \text{(ii)}$$

To find $a_0$ we integrate both sides of (ii) term-by-term from x = c to $x = c + 2\pi$, and obtain

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}[c + 2\pi - c] + 0 + 0 \qquad \text{(by article 6.1.4 (i) and (ii))}$$

Hence $$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \qquad \ldots\ldots \text{(iii)}$$

In order to find the coefficients $a_n$ we multiply each side of (ii) by cos nx and integrate the resulting series term-by-term from $x = c$ to $x = c + 2\pi$. This yields

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx = 0 + \pi a_n + 0$$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx, \quad n = 1, 2, 3, \ldots \qquad \ldots\ldots \text{(iv)}$$

Note that formula (iii) is obtained from (iv) by putting $n = 0$.

Finally, to find $b_n$ we multiply each side of (ii) by sin nx and integrate term by term from $x = c$ to $x = c + 2\pi$. Consequently, we have

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx = 0 + 0 + \pi b_n$$

$$\Rightarrow \quad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, 3, \ldots \qquad \ldots\ldots \text{(v)}$$

Thus the numbers $a_n$ and $b_n$ associated with f(x) can be calculated by means of formulae (iii), (iv) and (v) which are called **Euler's formulas.** Substituting these values of the coefficients in (ii) the Fourier series belonging to the function f(x) may be formally constructed.

**COROLLARY I** When $c = 0$, the formulae (iii), (iv) and (v) reduce respectively to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad \text{and}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \qquad \ldots\ldots \text{(vi)}$$

**COROLLARY II** When $c = -\pi$, the formulae (iii), (iv) and (v) become

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad \ldots\ldots \text{(vii)}$$

We now proceed to consider some examples to explain the behaviour of the Fourier series.

**EXAMPLE 1** Find the Fourier series representing the Identity function

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

**SOLUTION** Let

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots (1)$$

By Euler's formulae

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} 1\cdot\left(\frac{\sin nx}{n}\right)dx$$

$$= 0 - \frac{1}{n\pi}\cdot\left[-\frac{\cos nx}{n}\right]_0^{2\pi}$$

$$= \frac{1}{n^2\pi}\,[\cos 2n\pi - 1] \quad = \frac{1}{\pi n^2}\,(1-1) \quad = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} 1\cdot\left(-\frac{\cos nx}{n}\right)dx$$

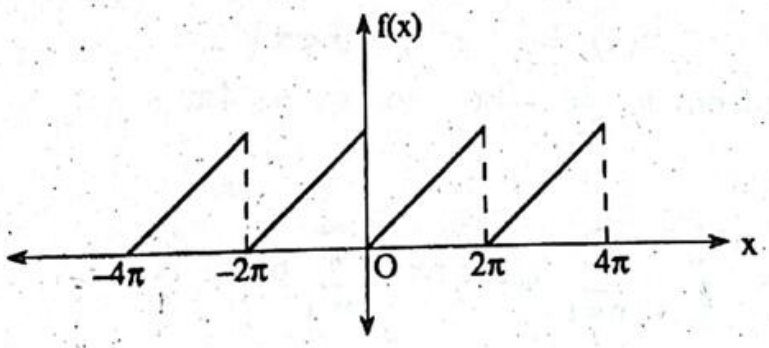
$$= \frac{1}{\pi}\left[-\frac{2\pi\cos 2n\pi}{n} - 0\right] + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{2\pi}$$

$$= -\frac{2}{n} \quad ; \quad \text{since } \cos 2n\pi = 1 \text{ and } \sin 2n\pi = 0$$

Substituting these values of $a_0$, $a_n$, $b_n$ in (1), we obtain

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right],$$

which is the required Fourier representation for the identify function in the given interval.




**EXAMPLE 2** Obtain the Fourier series for $e^x$ in the interval $-\pi < x < \pi$.

**SOLUTION** Let $f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ..... (1)

Then the Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\,[e^{\pi} - e^{-\pi}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x\sin nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{e^x\cos nx}{n^2}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^x\cos nx}{n^2}\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{e^x \cos nx}{n^2}\right]_{-\pi}^{\pi} - \frac{a_n}{n^2}$$

$$\left(1 + \frac{1}{n^2}\right)a_n = 0 + \frac{1}{\pi(n^2)}\left[e^x \cos nx\right]_{-\pi}^{\pi}$$

$$\therefore \quad a_n = \frac{1}{\pi(n^2+1)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right] = \frac{e^{\pi} - e^{-\pi}}{\pi(n^2+1)}\cos n\pi$$

$$= \frac{(-1)^n}{n^2+1}\left(\frac{e^{\pi} - e^{-\pi}}{\pi}\right), \quad \text{since } \cos n\pi = (-1)^n$$

$$\text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^x \sin nx - ne^x \cos nx\right]_{-\pi}^{\pi}$$

$$\left\{\because \quad \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right\}$$

$$= \frac{-n}{\pi(n^2+1)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right] = \frac{-n\cos n\pi}{(n^2+1)}\left(\frac{e^{\pi} - e^{-\pi}}{\pi}\right)$$

$$= \frac{n(-1)^{n+1}}{n^2+1}\left[\frac{e^{\pi} - e^{-\pi}}{\pi}\right]$$

Substituting the values of $a_0$, $a_n$, $b_n$ in (1), we get

$$e^x = \frac{e^{\pi} - e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right]$$

$$= \frac{2}{\pi}\sinh\pi\left[\frac{1}{2} - \left(\frac{\cos x}{2} + \frac{\cos 2x}{5} - \frac{\cos 3x}{10} \cdots\right) + \left(\frac{\sin x}{2} - \frac{2\sin 2x}{5} + \frac{3\sin 3x}{10} \cdots\right)\right]$$

**EXAMPLE 3** Find the Fourier series corresponding to

$f(x) = x + x^2$ in the interval $-\pi < x < \pi$

Also deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

**SOLUTION** Let

$$x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots (2)$$

Then $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[ \left| (x + x^2) \frac{\sin nx}{n} \right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1 + 2x) \frac{\sin nx}{n}\, dx \right]$$

$$= 0 - \frac{1}{n\pi} \left[ \left| (1 + 2x) \left(-\frac{\cos nx}{n}\right) \right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2 \left(-\frac{\cos nx}{n}\right) dx \right]$$

$$= \frac{1}{\pi n^2} [(1 + 2\pi) \cos n\pi - (1 - 2\pi) \cos(-n\pi)] + \frac{2}{\pi n^2} \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{4 \cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx\, dx$$

$$= \frac{1}{\pi} \left[ \left| (x + x^2) \left(-\frac{\cos nx}{n}\right) \right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1 + 2x) \left(\frac{-\cos nx}{n}\right) dx \right]$$

$$= \frac{1}{\pi} \left[ -(\pi + \pi^2) \frac{\cos n\pi}{n} + (-\pi + \pi^2) \frac{\cos n\pi}{n} \right]$$

$$+ \frac{1}{\pi} \left[ (1 + 2x) \cdot \frac{\sin nx}{n^2} + 2 \frac{\cos nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= -2 \frac{\cos n\pi}{n} = \frac{-2(-1)^n}{n}$$

Substituting these values of Fourier coefficients in (2), we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n}$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \quad \ldots\ldots (3)$$

Setting $x = \pi$ in (3), we have

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \ldots\ldots (4)$$

Putting $x = -\pi$ in (3), we have

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \ldots\ldots (5)$$

Adding (4) and (5), we obtain

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

## FUNCTION DEFINED IN TWO OR MORE SUB-RANGES

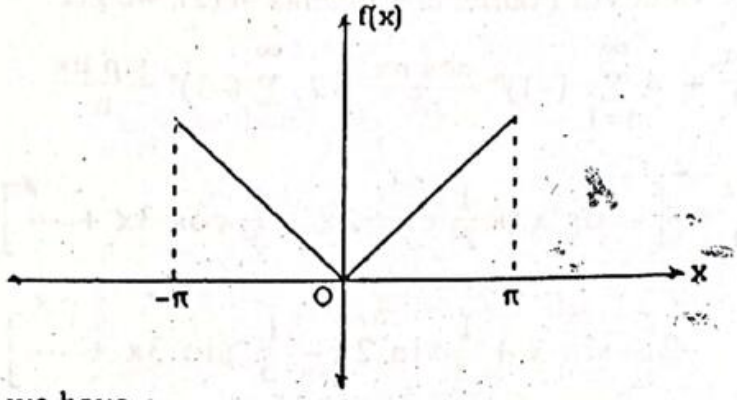
**EXAMPLE 4** Construct the Fourier series to represent

$$f(x) = -x, \quad -\pi < x \le 0$$

$$= x, \quad 0 < x < \pi$$

and deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

**SOLUTION**




In this case we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{-1}{\pi n^2} + \frac{\cos n\pi}{\pi n^2} + \frac{\cos n\pi}{\pi n^2} - \frac{1}{\pi n^2} = \frac{2}{\pi n^2}\,[\cos n\pi - 1]$$

$$= -\frac{4}{\pi n^2} \text{ or } 0 \text{ according as n is odd or even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} = 0$$

Hence the Fourier series corresponding to the given function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

**Deduction** Setting $x = 0$ in the above equation, we obtain

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

or $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

The same result will be obtained if we use $x = \pi$.

## 6.1.7 DIRICHLET CONDITIONS FOR CONVERGENCE

So for we have been representing a function in the form of its Fourier series under the assumption that f(x) converges uniformly. The detailed discussion of the conditions under which a function will be represented by its Fourier Series is beyond the scope of the book. So we state only the sufficient conditions for uniform convergence of a Fourier series, given by a German mathematician P.G.L. Dirichlet (1805–1859), in the following theorem;

**THEOREM** If a function f(x)

(i) is periodic, single valued and finite in $[-\pi, \pi]$
(ii) has a finite number of finite discontinuities
(iii) has only a finite number of maxima and minima in the interval,

then the representing Fourier Series converges to f(x) at every point at which it is continuous and converges to the average of the right hand and left hand limits of f(x) at each point of its discontinuity.

In other words the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges to

(a) f(x) if x is a point of continuity

(b) $\frac{f(x + 0) + f(x - 0)}{2}$ if x is a point of discontinuity,

where $f(x + 0)$ and $f(x - 0)$ are the right and left hand limits of f(x) at x and represent $\lim_{h \to 0} f(x + h)$ and $\lim_{h \to 0} f(x - h)$ respectively where $h > 0$. These are usually written $\lim_{h \to 0+} f(x + h)$ and $\lim_{h \to 0+} f(x - h)$ to emphasize that h approaches zero through positive values.

These conditions cover almost all the functions that normally we study in physical and engineering applications. Moreover it is clear from these conditions that a function need not be continuous in the given interval for its Fourier Series representation.

**EXAMPLE** If f(x) is defined by

$$f(x) = -\pi \quad \text{for} \quad -\pi < x < 0$$

$$= x \quad \text{for} \quad 0 < x < \pi,$$

find the Fourier series expansion for f(x). Also find the sum of the series for $x = 0$ and $x = \pm\pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

**SOLUTION**

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Now $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$

$$= \frac{1}{\pi}[-\pi x]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right]$$

$$= \begin{cases} 0 & \text{when n is even} \\ \frac{-2}{\pi n^2} & \text{when n is odd} \end{cases}$$

$\therefore \quad a_2 = 0 = a_4 = a_6 = \cdots$

$$a_1 = -\frac{2}{\pi}, \quad a_3 = -\frac{2}{\pi 3^2}, \quad a_5 = -\frac{2}{\pi 5^2}, \ \cdots$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi}{n}\cos n\pi\right] + \frac{1}{\pi}\left[-\frac{\pi \cos n\pi}{n} + 0\right]$$

$$= \frac{1}{n}(1 - 2\cos n\pi)$$

$$b_1 = 3, \ b_2 = -\frac{1}{2}, \ b_3 = +\frac{3}{3} \text{ etc.}$$

Therefore $f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

$$+\left(3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \quad \ldots (1)$$

Now $x = 0, \pm\pi$ are points of discontinuity.

Putting $x = 0$ in (1), we obtain

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \quad \ldots\ldots (2)$$

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

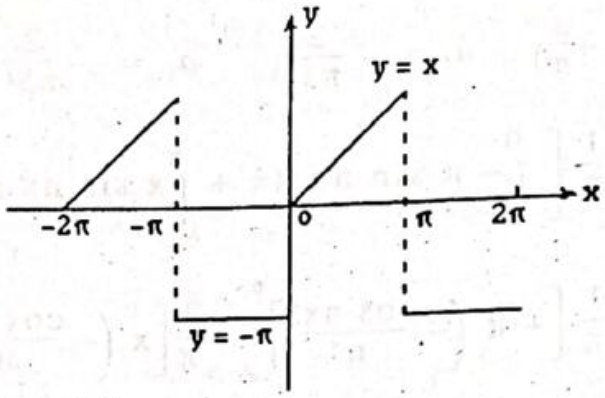
$f(0-0) = -\pi$ and $f(0+0) = 0$

$\therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

Hence (2) takes the form

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

which implies that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$




Also when $x = \pi$,

the value of the series $= \dfrac{f(\pi +) + f(\pi -)}{2}$

$= \dfrac{-\pi + \pi}{2} = 0$

Similarly when $x = -\pi$, the value of the series is zero.

## EXERCISE 6.1

**1.** Obtain the Fourier series representing

$$f(x) = x \ , \quad -\pi < x < \pi$$

**2.** Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

**3.** Find the Fourier series expansion of the step-function f(x) which is periodic with period $2\pi$ and is defined in $-\pi < x < \pi$ by

$$f(x) = \begin{cases} -1 & , \quad -\pi < x < 0 \\ 0 & , \quad x = 0 \\ +1 & , \quad 0 < x < \pi \end{cases}$$

**4.** Expand the function

$$f(x) = 1 \ , \quad 0 < x < \pi$$
$$= 2 \ , \quad \pi < x < 2\pi$$

in a Fourier series and examine the series at the points of discontinuity.

**5.** Develop f(x) in a Fourier series if defined as:

$$f(x) = 0 \ , \quad -\pi < x \leq 0$$
$$= 1 \ , \quad 0 < x \leq \pi$$

6. Find the Fourier series for the function f(x) defined as

$$f(x) = \begin{cases} x + \pi & , \quad 0 \le x \le \pi \\ -x - \pi & , \quad -\pi \le x < 0 \end{cases}$$

and $f(x + 2\pi) = f(x)$

7. Obtain Fourier series representing the function

$$f(x) = 0 \quad , \quad -\pi \le x \le 0$$
$$= \sin x \quad , \quad 0 \le x \le \pi$$

8. Find the Fourier series corresponding to the function f(x) defined by

$$f(x) = x \quad , \quad 0 \le x \le \pi$$
$$= 2\pi - x \quad , \quad \pi \le x \le 2\pi$$

9. Obtain Fourier series expansion for f(x) defined as follows:

$$f(x) = x + \frac{\pi}{2} \quad , \quad -\pi < x \le 0$$
$$= \frac{\pi}{2} - x \quad , \quad 0 \le x < \pi$$

10. Find the Fourier series to represent

$$f(x) = \frac{1}{4}(\pi - x)^2 \quad , \quad 0 < x < 2\pi$$

11. Find the Fourier series expansion for f(x), if $f(x) = x^2$ , $0 < x < 2\pi$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

12. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$ and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

13. Expand $f(x) = x \sin x$ , $0 \le x \le 2\pi$, in a Fourier series.

14. Expand the following functions in a Fourier Series.

(i) $f(x) = x$ , if $-\pi < x \le 0$

$= 2x$ , if $0 \le x \le \pi$.

(ii) $f(x) = x^2$ , $-\pi \le x \le \pi$.

## 6.2 FUNCTIONS HAVING ARBITRARY PERIODS

In many practical problems we require Fourier series representation of a function over an interval of length different from $2\pi$. In such cases the interval of arbitrary length is converted into one of length $2\pi$ by effecting a change in the independent variable proportionally.

Let a function f(x) with period $2\ell$ be defined for the interval $-\ell < x < \ell$, where $\ell$ is any positive number. In order to change the interval into one of length $2\pi$, we introduce the transformation

$$x = \frac{\ell}{\pi} t$$

Then the function f(x) defined in $-\ell < x < \ell$ changes to the function $f\left(\frac{\ell}{\pi} t\right)$ or g(t) valid for $-\pi < t < \pi$ having period $2\pi$, since

$$g(t + 2\pi) = f\left[\frac{\ell}{\pi}(t + 2\pi)\right] = f\left(\frac{\ell}{\pi} t + 2\ell\right) = f\left(\frac{\ell}{\pi} t\right) = g(t).$$

Hence the Fourier series corresponding to g(t) is

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad \ldots\ldots \text{(i)}$$

with Euler's formulae for Fourier coefficients given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt \, dt, \quad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt \, dt, \quad n = 1, 2, 3, \cdots$$

To find the Fourier series for f(x) in $-\ell < x < \ell$, we replace t by $\frac{\pi}{\ell} x$ in (i) and obtain

$$f(x) = g\left(\frac{\pi}{\ell} x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}\right) \quad \ldots\ldots \text{(ii)}$$

with the Euler's formulae for Fourier coefficients as

$$a_n = \frac{1}{\pi} \int_{-\ell}^{\ell} g\left(\frac{\pi x}{\ell}\right) \cos \frac{n\pi x}{\ell} \cdot \frac{\pi}{\ell} dx = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \; n = 0,1,2,\cdots$$

$$b_n = \frac{1}{\pi} \int_{-\ell}^{\ell} g\left(\frac{\pi x}{\ell}\right) \sin \frac{n\pi x}{\ell} \cdot \frac{\pi}{\ell} dx = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx, \; n = 1,2,3,\cdots$$

**NOTE** If we use the limits 0 and $2\ell$ instead of $-\ell$ and $\ell$, then the corresponding Euler's formulae are

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad n = 0, 1, 2, \ldots\ldots$$

and $$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \quad n = 1, 2, 3, \ldots\ldots$$

**EXAMPLE 1** Expand into Fourier series :

$$f(x) = x^2, \; -1 < x < 1$$

**SOLUTION** Let $x = \frac{2}{2\pi}t = \frac{t}{\pi}$ in $-\pi < t < \pi$

so that $g(t) = \frac{t^2}{\pi^2}$

Let $g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$ ..... (1)

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t^2}{\pi^2}\,dt = \frac{1}{\pi^3}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t^2}{\pi^2}\cos nt\,dt = \frac{1}{\pi^3}\left[t^2\cdot\frac{\sin nt}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi^3}\int_{-\pi}^{\pi}2t\left(\frac{\sin nt}{n}\right)dt$$

$$= 0 - \frac{2}{n\pi^3}\left[t\cdot\left(\frac{-\cos nt}{n}\right)\right]_{-\pi}^{\pi} + \frac{2}{n\pi^3}\int_{-\pi}^{\pi}\left(\frac{-\cos nt}{n}\right)dt$$

$$= \frac{2}{n^2\pi^3}[2\pi\cos n\pi] - \frac{2}{n^2\pi^3}\left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t^2}{\pi^2}\sin nt\,dt = \frac{1}{\pi^3}\left[t^2\left(\frac{-\cos nt}{n}\right) - (2t)\left(-\frac{\sin nt}{n^2}\right) + 2\left(\frac{\cos nt}{n^3}\right)\right]_{-\pi}^{\pi}$$

{applying generalised rule of integration by parts see (ix) of Art.6.1.4}

$$= \frac{1}{\pi^3}\left[\frac{-\pi^2}{n}(\cos n\pi - \cos n\pi) + 0 + \frac{2}{n^3}(\cos n\pi - \cos n\pi)\right] = 0$$

Substituting in (1), we have

$$g(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt$$

Replacing t by $\pi x$, the Fourier series for f(x) is

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

**EXAMPLE 2** Prove, by Fourier series method, that

$$\frac{l}{2} - x = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}, \quad 0 < x < l.$$

**SOLUTION** Here length of the interval $2c = l \Rightarrow c = l/2$

Let $f(x) = \frac{l}{2} - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l/2} + b_n \sin \frac{n\pi x}{l/2} \right)$ ... (1)

see Art. 6.2 (ii)

Then $$a_0 = \frac{1}{\frac{l}{2}} \int_0^l f(x)\,dx = \frac{2}{l} \int_0^l \left( \frac{l}{2} - x \right) dx$$

$$= \frac{2}{l} \left[ \frac{lx}{2} - \frac{x^2}{2} \right]_0^l = 0$$

$$a_n = \frac{1}{\frac{l}{2}} \int_0^l f(x) \cos \frac{n\pi x}{l/2}\,dx$$

$$= \frac{2}{l} \int_0^l \left( \frac{l}{2} - x \right) \cos \frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l} \left[ \left| \left( \frac{l}{2} - x \right) \frac{l}{2n\pi} \sin \frac{2n\pi x}{l} \right|_0^l - \frac{l}{2n\pi} \int_0^l (-1) \sin \frac{2n\pi x}{l}\,dx \right]$$

$$= \frac{2}{l} \left[ 0 + \frac{l}{2n\pi} \frac{l}{2n\pi} \cos \frac{2n\pi x}{l} \right]_0^l$$

$$= \frac{2}{\ell}\,\frac{\ell^2}{4n^2\pi^2}\quad(\cos 2n\pi - 1)$$

$$= \frac{\ell}{2n^2\pi^2}\,(1-1) \;=\; 0$$

$$b_n \;=\; \frac{1}{\ell/2}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell/2}\,dx$$

$$= \frac{2}{\ell}\int_0^{\ell}\left(\frac{\ell}{2} - x\right)\sin\frac{2n\pi x}{\ell}\,dx$$

$$= \frac{2}{\ell}\left[\left|\left(\frac{\ell}{2}-x\right)\left(-\frac{\ell}{2n\pi}\cos\frac{2n\pi x}{\ell}\right)\right|_0^{\ell} - \int_0^{\ell}(-1)\left(-\frac{\ell}{2n\pi}\cos\frac{2n\pi x}{\ell}\right)dx\right]$$

$$= \frac{2}{\ell}\left[\frac{\ell}{2}\,\frac{\ell}{2n\pi}\cos 2n\pi + \frac{\ell}{2}\cdot\frac{\ell}{2n\pi} - \left\{\left(\frac{\ell}{2n\pi}\right)\left(\frac{\ell}{2n\pi}\right)\sin\frac{2n\pi x}{\ell}\right\}_0^{\ell}\right]$$

$$= \frac{2}{\ell}\left[\frac{\ell^2}{2n\pi}\right] = \frac{\ell}{n\pi}$$

Thus from (1) the Fourier series is

$$f(x) \;=\; \frac{a_0}{2} + a_1\cos\frac{\pi x}{\ell/2} + a_2\cos\frac{2\pi x}{\ell/2} + \cdots$$

$$+ b_1\sin\frac{\pi x}{\ell/2} + b_2\sin\frac{2\pi x}{\ell/2} + \cdots$$

$$\text{or}\quad \frac{\ell}{2} - x \;=\; \frac{\ell}{\pi}\sin\frac{2\pi x}{\ell} + \frac{\ell}{2\pi}\sin\frac{4\pi x}{\ell} + \frac{\ell}{3\pi}\sin\frac{6\pi x}{\ell} + \cdots$$

$$= \frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{\ell}.$$

**EXAMPLE 3** Obtain the Fourier series for

$$f(x) \;=\; \begin{cases} x, & \text{for } -1 < x \le 0 \\ x+2, & \text{for } 0 < x \le 1. \end{cases}$$

**SOLUTION** The required series is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) \quad \ldots\ldots (1)$$

where $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$$= \int_{-1}^{0} x\,dx + \int_{0}^{1} (x+2)\,dx = 2,$$

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x\,dx$$

$$= \int_{-1}^{0} f(x) \cos n\pi x\,dx + \int_{0}^{1} f(x) \cos n\pi x\,dx$$

$$= \int_{-1}^{0} x \cos n\pi x\,dx + \int_{0}^{1} (x+2) \cos n\pi x\,dx.$$

$$= 2\left[\frac{\sin n\pi x}{n\pi}\right]_0^1 = \frac{2}{n\pi} \cdot \sin n\pi = 0,$$

and $b_n = \int_{-1}^{1} f(x) \sin n\pi x\,dx$

$$= \int_{-1}^{0} x \sin n\pi x\,dx + \int_{0}^{1} (x+2) \sin n\pi x\,dx$$

$$= 2\int_{0}^{1} x \sin n\pi x\,dx + 2\int_{0}^{1} \sin n\pi x\,dx$$

$$= 2\left[-x \frac{\cos n\pi x}{n\pi} + \frac{1}{n^2\pi^2} \sin n\pi x\right]_0^1 - 2\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{2\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi}$$

$$= -\frac{4\cos n\pi}{n\pi} + \frac{2}{n\pi}$$

$$= \frac{2}{n\pi} - \frac{4}{n\pi}(-1)^n$$

$$= \frac{2}{n\pi}\,[1 - 2(-1)^n].$$

Hence, $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,[1 - 2(-1)^n]\sin n\pi x$

$$= 1 + \frac{2}{\pi}\left[3\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\cdot 3\cdot \sin 3\pi x - \frac{1}{4}\cdot \sin 4\pi x + \frac{1}{5}\cdot 3\cdot \sin 5\pi x - \frac{1}{6}\sin 6\pi x + \frac{1}{7}\cdot 3\cdot \sin 7\pi x - \cdots\right]$$

## EXERCISE 6.2

1. Expand into Fourier series the function :
   (i) $f(x) = x,\ 0 < x < 1.$ (ii) $f(x) = 2x + 1,\ 0 < x < 2.$

2. Expand f(x) in Fourier series in the interval (–2, 2) when
$$f(x) = 0,\ -2 < x < 0$$
$$= 1,\ 0 < x < 2.$$

3. Find a Fourier series for
$$f(x) = x^2,\ 1 < x < 2$$

4. Expand $f(x) = x - x^2$ as a Fourier series in the interval (–1, 1).

5. Develop f(x) in a Fourier series in the interval (0, 2) if
$$f(x) = x,\ 0 < x < 1$$
$$= 0,\ 1 < x < 2.$$

6. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.

7. Expand into Fourier series
$$f(x) = 0,\ -2 < x < -1$$
$$= k,\ -1 < x < 1$$
$$= 0,\ 1 < x < 2$$

8. Find the Fourier series for the function defined by
$$f(x) = x + 1,\ -1 \le x < 0$$
$$= x - 1,\ 0 \le x \le 1$$

**9.** Obtain Fourier series for the function defined below :

$$f(x) = \pi x, \quad 0 \le x \le 1$$
$$= \pi(2 - x), \quad 1 \le x \le 2.$$

**10.** Expand into Fourier series:

$$f(x) = 1 - |x|, \quad -1 \le x \le 1.$$

**11.** Find the Fourier series corresponding to the function f(x) defined in (–2, 2) as follows;

$$f(x) = 2, \quad -2 \le x \le 0$$
$$= x, \quad 0 < x < 2.$$

**12.** Obtain Fourier series for f(x) of period $2l$ defined as follows;

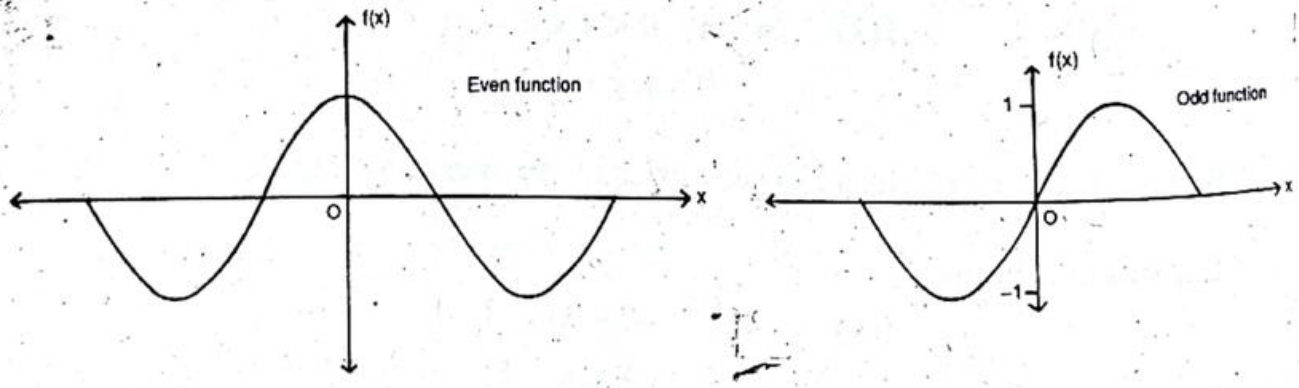
$$f(x) = l - x, \quad 0 < x \le l$$
$$= 0, \quad l \le x < 2l.$$

## 6.3 EVEN AND ODD FUNCTIONS

The knowledge of even and odd functions helps us in reducing our calculations of Fourier coefficients while expanding a function in Fourier series.

### 6.3.1 DEFINITIONS

A function f(x) is said to be **even** if $f(-x) = f(x)$ e.g. cos x or |x| is an even function. Any even power of x is also an even function. The graph of an even function is symmetrical with respect to y-axis. Moreover, the area under such




a curve from –c to c is double the area from 0 to c.

$$\text{i.e.} \quad \int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$$

For $\int_{-c}^{c} f(x)\,dx = \int_{-c}^{0} f(x)\,dx + \int_{0}^{c} f(x)\,dx = I_1 + I_2$ ..... (i)

In $I_1$, putting $x = -t$ so that $dx = -dt$, we have

$$I_1 = \int_{-c}^{0} f(x)\,dx = \int_{c}^{0} f(-t)\,(-dt) = -\int_{c}^{0} f(t)\,dt$$

$$= \int_{0}^{c} f(t)\,dt = \int_{0}^{c} f(x)\,dx, \text{ because t and x are dummy variables.}$$

Hence (i) becomes

$$\int_{-c}^{c} f(x)\,dx = \int_{0}^{c} f(x)\,dx + \int_{0}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$$

A function f(x) is said to be **odd** if $f(-x) = -f(x)$. For example, any odd power of x is an odd function of x and so are sin x or sin 2x. The graph of such a function is symmetric with respect to the origin. The area under such a curve from –c to c is zero.

i.e. $$\int_{-c}^{c} f(x)\,dx = 0$$

For $\int_{-c}^{c} f(x)\,dx = \int_{-c}^{0} f(x)\,dx + \int_{0}^{c} f(x)\,dx$

$$= \int_{c}^{0} f(-t)\,(-dt) + \int_{0}^{c} f(x)\,dx, \text{ by putting } x = -t \text{ in the first integral.}$$

$$= \int_{c}^{0} [-f(t)]\,(-dt) + \int_{0}^{c} f(x)\,dx$$

$$= -\int_{0}^{c} f(x)\,dx + \int_{0}^{c} f(x)\,dx$$

$$= 0$$

It is obvious from the definitions that sums of even (odd) functions are even (odd). The product of two even or two odd functions is even.

Let f(x) and g(x) be both even functions. If $\phi(x) = f(x)\,g(x)$, then

$$\phi(-x) = f(-x)\,g(-x) = f(x)\,g(x) = \phi(x)$$

That is, the product of two even functions is an even function.

However, the product of an even and an odd function is odd. Let f(t) be an even and g(t) be an odd function. If $h(t) = f(t)\,g(t)$, then

$$h(-t) = f(-t)\,g(-t) = f(t)\,[-g(t)] = -f(t)\,g(t) = -h(t)$$

A function may be neither even nor odd. e.g., $e^x$, $2x + x^2$, $\sin x + \cos x$ etc. But we can express a given function as the sum of an even function and of an odd function. For example,

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \cosh x + \sinh x$$

$$= \text{Even function} + \text{Odd function.}$$

## 6.3.2 FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know from Article 6.2 that if f(x) is a function defined in $(-l, l)$ the corresponding Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \ldots\ldots \text{(ii)}$$

having Fourier coefficients

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx\ ,\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

**CASE I** When f(x) is an **even function**, then

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

Since f(x) and $\cos\frac{n\pi x}{l}$ are both even functions, their product $f(x)\cos\frac{n\pi x}{l}$ is also an even function. Therefore,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

Again, since f(x) is even and $\sin\frac{n\pi x}{l}$ is odd, their product $f(x)\sin\frac{n\pi x}{l}$ will be an odd function. Therefore,

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0$$

Hence, if f(x) is an even function, then its Fourier expansion (ii) contains cosine terms only.

**CASE II** If f(x) is an **odd function**, then

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx = 0$$

Since $\cos\frac{n\pi x}{\ell}$ is an even function, therefore the product $f(x)\cos\frac{n\pi x}{\ell}$ is odd and consequently

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = 0$$

Moreover, $\sin\frac{n\pi x}{\ell}$ being odd, the product $f(x)\sin\frac{n\pi x}{\ell}$ will be an even function and as a result

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Thus we have proved that the Fourier expansion of an odd function f(x) defined in $(-\ell, \ell)$ contains sine terms only.

**COROLLARY** If f(x) is an even function defined in $-\pi < x < \pi$, then the coefficients in the corresponding Fourier series are

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \quad b_n = 0$$

In case the function f(x) defined in $-\pi < x < \pi$ is odd, then the Euler's formulae are given by

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

**EXAMPLE 1** Find the Fourier expansion for the function:

$$f(x) = \sin kx, \quad -\ell < x < \ell$$

**SOLUTION** Since $f(-x) = -\sin kx = -f(x)$,

therefore, the given function is odd and hence the Fourier expansion is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

where $b_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{2}{\ell}\int_{0}^{\ell} \sin kx \sin\frac{n\pi x}{\ell}\,dx$

$$= \frac{1}{\ell}\int_0^{\ell}\left[\cos\left(\frac{n\pi}{\ell} - k\right)x - \cos\left(\frac{n\pi}{\ell} + k\right)x\right]dx$$

$$= \frac{1}{\ell}\left[\frac{\ell}{n\pi - k\ell}\sin\left(\frac{n\pi}{\ell} - k\right)x - \frac{\ell}{n\pi + k\ell}\sin\left(\frac{n\pi}{\ell} + k\right)x\right]_0^{\ell}$$

$$= \frac{1}{n\pi - k\ell}\sin(n\pi - k\ell) - \frac{1}{n\pi + k\ell}\sin(n\pi + k\ell)$$

$$\therefore \quad b_1 = \frac{1}{\pi - k\ell}\sin k\ell + \frac{1}{\pi + k\ell}\sin k\ell = \frac{2\pi}{\pi^2 - k^2\ell^2}\sin k\ell$$

$$b_2 = \frac{-1}{2\pi - k\ell}\sin k\ell - \frac{1}{2\pi + k\ell}\sin k\ell = \frac{-4\pi}{4\pi^2 - k^2\ell^2}\sin k\ell$$

$$b_3 = \frac{1}{3\pi - k\ell}\sin k\ell + \frac{1}{3\pi + k\ell}\sin k\ell = \frac{6\pi}{9\pi^2 - k^2\ell^2}\sin k\ell$$

and so on.

Hence the required Fourier expansion is

$$f(x) = 2\pi \sin k\ell\left[\frac{1}{\pi^2 - k^2\ell^2}\sin\frac{\pi x}{\ell} - \frac{2}{2^2\pi^2 - k^2\ell^2}\sin\frac{2\pi x}{\ell} + \frac{3}{3^2\pi^2 - k^2\ell^2}\sin\frac{3\pi x}{\ell} + \ldots\right]$$

**EXAMPLE 2** Expand $f(x) = |\cos x|$ in a Fouries series in the interval $(-\pi, \pi)$.

**SOLUTION** Since the function $f(x)$ is an even function, therefore the corresponding Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots\ldots (1)$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)dx\right]$$

$$\left\{\because \begin{array}{l}\cos x \text{ is } -\text{ve when } \pi/2 < x < \pi,\\ \text{Hence in } (\pi/2, \pi),\ |\cos x| = -\cos x\end{array}\right\}$$

$$= \frac{2}{\pi}\left[|\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right] = \frac{4}{\pi}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi/2} 2\cos x \cos nx \, dx - \int_{\pi/2}^{\pi} 2\cos x \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\}\, dx \right.$$

$$\left. - \int_{\pi/2}^{\pi} \{\cos(n+1)x + \cos(n-1)x dx\}\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\pi/2}\right.$$

$$\left. - \left|\frac{\sin(n+1)\pi}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\}\right.$$

$$\left. + \left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2\cos n\pi/2}{\pi}\left(\frac{-2}{n^2-1}\right)$$

$$= \frac{-4 \cos n\pi/2}{\pi(n^2 - 1)} \qquad (n \neq 1)$$

For $n = 1$, $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right] = 0$

Hence (1) yields $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right]$

## EXERCISE 6.3

**1.** Express the function $f(x) = x$, $-\pi < x < \pi$ in a Fouries series.

**2.** Expand into Fouries series

$f(x) = \cos x \quad , \quad \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

**3.** Obtain a Fouries series for

$f(x) = x^3 \quad , \quad -\pi < x < \pi$

**4.** Expand in a Fouries series

$f(x) = |x| \quad , \quad -1 < x < 1$

**5.** Find a Fouries series for $f(x) = 1 - x^2$, $-1 \leq x \leq 1$

**6.** Express $f(x) = \cos px$, $-\pi < x < \pi$, where p is a fraction, as a Fourier series.

**7.** Find the Fourier series to represent the function

$f(x) = |\sin x| \quad , \quad -\pi < x < \pi$

**8.** Find the Fourier series for $f(x)$ if

$$f(x) = -\frac{1}{2} \quad , \quad -\pi < x < 0$$
$$= \frac{1}{2} \quad , \quad 0 < x < \pi$$

and $f(x + 2\pi) = f(x)$

**9.** Obtain Fourier series for the function $f(x)$ given by

$$f(x) = 1 + \frac{2x}{\pi} \quad , \quad -\pi \leq x \leq 0$$
$$= 1 - \frac{2x}{\pi} \quad , \quad 0 \leq x \leq \pi.$$

and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

10. By Fourier series, for $-\pi < x < \pi$ show that

$$x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} n \sin nx$$

## 6.4 HALF RANGE SERIES

Sometimes we are required to expand a function into Fourier series defined over only a half period $l$, say in $0 < x < l$ instead of an interval of length $2l$. Moreover, we may be forced by the conditions of the problem to represent a given function in a series of sines alone or a series of cosines alone, in which case we can make use of our discussion of Article 6.3.2 about even and odd functions. As it is immaterial whatever the function may be outside the range $0 < x < l$, we extend the function to cover the full range $-l < x < l$ such that the new function is either even or odd.

### (i) *Fourier sine series*

If we desire to expand a given function f(x) as a series of sines in half-range $0 < x < l$, then we extend the function reflecting it in the origin so that $f(-x) = -f(x)$ that is, we construct a function $\phi(x)$ which is identical with f(x) for $0 < x < l$ but equal to $-f(-x)$ for $-l < x < 0$:

$$\phi(x) = f(x), \quad 0 < x < l$$

$$= -f(-x), \quad -l < x < 0$$

Then the extended function $\phi(x)$ is an odd function of x defined in the interval $-l < x < l$ and consequently will have a Fourier series involving sine terms alone given by

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

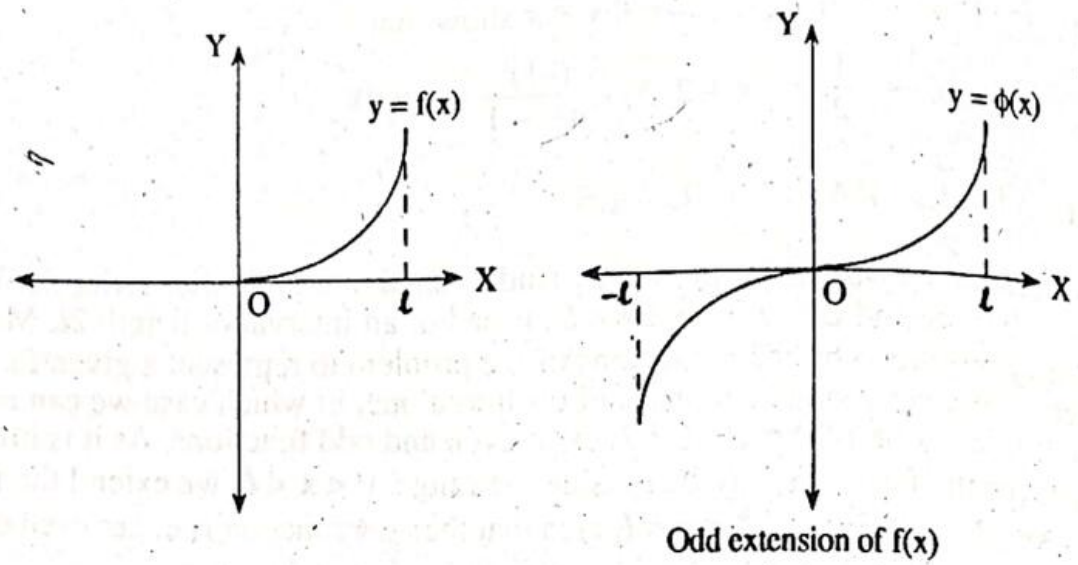
where $$b_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \sin \frac{n\pi x}{l} \, dx = \frac{2}{l} \int_{0}^{l} \phi(x) \sin \frac{n\pi x}{l} \, dx$$

But $\phi(x)$ is identical with f(x) in $0 < x < l$, therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \qquad \ldots\ldots \text{(i)}$$

where $$b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 1, 2, \ldots$$

The series (i) is known as Fourier sine series of f(x) on the interval $0 < x < l$.




Odd extension of f(x)

### (ii) *Fourier cosine series*

If we are interested in expressing a function f(x) defined in the interval $0 < x < l$ as a Fourier series containing cosine terms only, then we create a new function $\phi(x)$ such that

$$\phi(x) = f(x) \quad , \quad 0 < x < l$$

$$= f(-x) \quad , \quad -l < x < 0$$

Thus extended function $\phi(x)$ is an even function of x in $-l < x < l$ and its graph will be symmetrical about the y–axis. Hence by Article 6.3.2 its expansion will yield the Fourier cosine series:

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

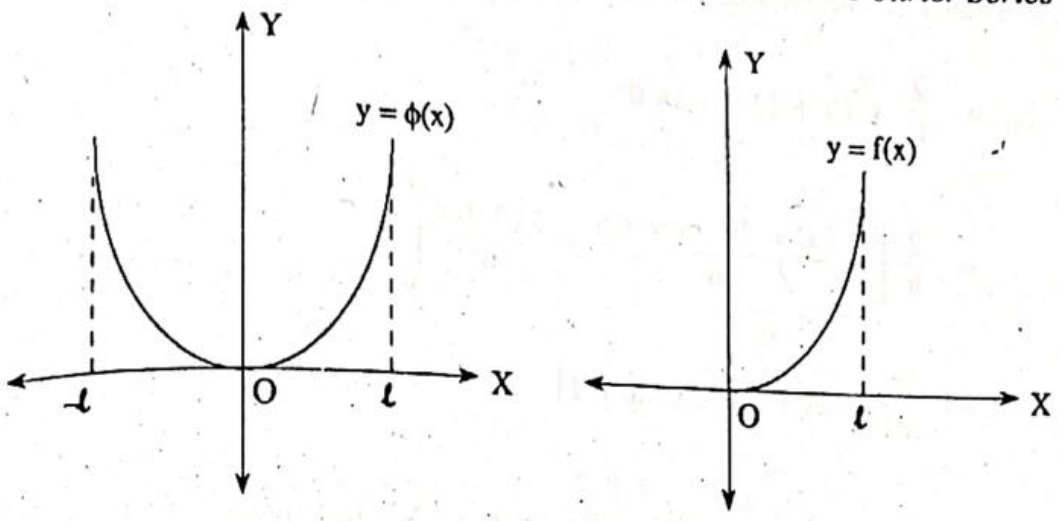
where $$a_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \int_{0}^{l} \phi(x) \cos \frac{n\pi x}{l} dx$$

But $\phi(x) = f(x)$ in the half-range $0 < x < l$. Therefore, in this interval f(x) will be represented by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \qquad \ldots\ldots \text{(ii)}$$

where $$a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} dx, \; n = 0, 1, 2, \ldots$$

Series (ii) involving cosines only is also called half-range expansion.




Even Extension of f(x)

**NOTE** Instead of $0 < x < l$ if $f(x)$ is defined in $0 < x < \pi$, then it may be

(i) represented by the series of sines

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx, \quad n \geq 1$$

(ii) expressed by the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad n \geq 0.$$

**EXAMPLE 1** Expand $f(x) = x + 1, 0 < x < \pi$ in a Fourier sine series.

**SOLUTION** Let us define $\phi(x)$ as follows:

$$\phi(x) = -f(-x) = -(-x + 1) = x - 1, -\pi < x < 0,$$

and $$\phi(x) = f(x) = x + 1, 0 < x < \pi.$$

Here $\phi(-x) = -f(x) = -\phi(x)$. Therefore, $\phi(x)$ is an odd function.

$\therefore$ $a_n = 0$

and $b_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin nx\,dx$

$$= \frac{2}{\pi}\left[-\frac{(x+1)\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n}\left[-(\pi+1)\cos n\pi + 1\right]$$

Hence the required series of sines is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x) = \frac{2}{\pi}\left[(\pi+2)\sin x - \frac{\pi}{2}\sin 2x + \frac{1}{3}(\pi+2)\sin 3x - \frac{\pi}{4}\sin 4x + \ldots\right]$$

**EXAMPLE 2** Find the Fourier half-range

(i) sine series

(ii) cosine series

for the function defined by $f(x) = x^2,\ 0 < x < \pi$.

**SOLUTION**

(i) Let $f(x) = x^2 = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right]$$

$$= \begin{cases} \frac{2}{\pi}\left[\frac{\pi^2}{n} - \frac{4}{n^3}\right] \text{ when n is odd} & \ldots\ldots (1) \\ \frac{2}{\pi}\left[-\frac{\pi^2}{n}\right] = -\frac{2\pi}{n} \text{ when n is even} & \ldots\ldots (2) \end{cases}$$

Putting $n = 1, 3, 5 \ldots$ in (1), we obtain

$$b_1 = \frac{2}{\pi}\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right),\ b_3 = \frac{2}{\pi}\left(\frac{\pi^2}{3} - \frac{4}{3^3}\right), \ldots$$

Setting $n = 2, 4, 6, \ldots$ in (2), we get

$$b_2 = -\frac{2\pi}{2},\ b_4 = -\frac{2\pi}{4}, \ldots$$

Therefore, the required Fourier sine series is

$$x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x + \ldots\right]$$

$$-2\pi\left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots\right)$$

(ii). Let $f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ..... (3)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2,$

and $a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] = \frac{4\cos n\pi}{n^2} \qquad \ldots\ldots (4)$$

Putting $n = 1, 2, 3, \ldots$ in the expression (4) for $a_n$ we get

$$a_1 = -\frac{4}{1^2},\ a_2 = \frac{4}{2^2},\ a_3 = -\frac{4}{3^2}, \ldots$$

Therefore, series (3) implies

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x \ldots\right)$$

**EXAMPLE 3** Represent as Fourier sine series:

$$f(x) = 1 \quad , \quad 0 < x < l/2$$

$$= 0 \quad , \quad l/2 < x < l$$

**SOLUTION** Let the sine series be $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$= \frac{2}{l} \left[ \int_0^{l/2} 1 . \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} 0 . \sin \frac{n\pi x}{l} \, dx \right]$$

$$= \frac{2}{l} \int_0^{l/2} \sin \frac{n\pi x}{l} \, dx = \frac{2}{l} \left[ - \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right]_0^{l/2}$$

$$= -\frac{2}{n\pi} \left[ \cos \frac{n\pi x}{l} \right]_0^{l/2} = \frac{-2}{n\pi} \left[ \cos \frac{n\pi}{2} - 1 \right] = \frac{2}{n\pi} \left( 1 - \cos \frac{n\pi}{2} \right)$$

$$\therefore \quad b_1 = \frac{2}{\pi}, \quad b_2 = \frac{2}{2\pi} (1 - \cos \pi) = \frac{2}{\pi},$$

$$b_3 = \frac{2}{3\pi} \left( 1 - \cos \frac{3\pi}{2} \right) = \frac{2}{3\pi}, \quad b_4 = 0, \text{ etc.}$$

$$\therefore \quad f(x) = \frac{2}{\pi} \left[ \sin \frac{\pi x}{l} + \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \frac{1}{5} \sin \frac{5\pi x}{l} + \cdots \right]$$

**EXAMPLE 4** Prove that for all values of x between $\frac{-\pi}{2}$ and $+\frac{\pi}{2}$,

$$x = \frac{4}{\pi} \left( \sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} \cdots \right)$$

**SOLUTION** We have $-\frac{\pi}{2} < x < \frac{\pi}{2}$ or $0 < x + \frac{\pi}{2} < \pi$

Let $x + \frac{\pi}{2} = t$. Then $0 < t < \pi$.

Now we expand $f(t) = t$, $0 < t < \pi$ in a cosine series. Corresponding Fourier co-efficients are

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \pi$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\frac{t\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - 1\right]$$

Hence $$t = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right]$$

i.e., $$x + \frac{\pi}{2} = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos\left(\frac{\pi}{2} + x\right) + \frac{1}{3^2}\cos 3\left(\frac{\pi}{2} + x\right) + \frac{1}{5^2}\cos 5\left(\frac{\pi}{2} + x\right) + \cdots\right]$$

or $$x = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} \cdots\right]$$

which gives the required result.

## EXERCISE 6.4

1. Given a constant k, show that in the half-range $0 < x < \pi$,

$$k = \frac{4}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}k$$

2. Find the half-range Fourier cosine series for the function

$$f(x) = x,\ 0 < x < \pi$$

3. Obtain a half-range cosine series for

$$f(x) = 2x - 1 \quad , \quad 0 < x < 1$$

Hence show that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

4. Find the half-range sine series for the function defined as:

$$f(x) = e^x \quad , \quad 0 < x < 1$$

5. Expand $f(x) = e^x$ as Fourier cosine series for $0 < x < 1$.

6. Represent by Fourier sine and cosine series the function defined by

$$f(x) = 0 \quad , \quad 0 \le x < \pi/2$$
$$= 1 \quad , \quad \pi/2 \le x \le \pi$$

7. Find the half-range sine and cosine series for the function

$$f(x) = x \quad , \quad 0 < x \le \frac{\pi}{2}$$
$$= \pi - x \quad , \quad \frac{\pi}{2} < x < \pi$$

8. Express sinx as a half-range cosine series in $0 < x < \pi$

and deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \pi/4$

9. Expand $f(x) = \frac{1}{4} - x \quad , \quad 0 < x < \frac{1}{2}$
$= x - \frac{3}{4} \quad , \quad \frac{1}{2} < x < 1$

as a Fourier sine series.

10. Expand $x(\pi - x)$ in half-range sine series when $0 < x < \pi$.

11. Expand $f(x) = \pi - x$ into

(i) Fourier series for $-\pi < x < \pi$

(ii) Fourier sine series for $0 < x < \pi$

(iii) Fourier cosine series for $0 \le x \le \pi$

12. Obtain Fourier sine series if $f(x) = \cos\left(\frac{\pi x}{p}\right)$, $0 \le x \le p$.

13. Express $f(x) = \sin\left(\frac{\pi x}{\ell}\right)$, $0 < x < \ell$, by a Fourier cosine series.

14. Let $f(x) = \cos x$ for $0 < x < \pi$. If $f(x)$ is odd and periodic with period $2\pi$, show that the Fourier series for $f(x)$ is given by

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{m \sin 2mx}{(4m^2 - 1)}.$$

7

# DIFFERENTIAL EQUATIONS - I

## 7.1 SOME BASIC DEFINITIONS

### 7.1.1 DEFINITION

An equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables, is called a **differential equation.**

### 7.1.2 CLASSIFICATION

An equation involving only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, is said to be an **ordinary differential equation.**

For example,

$$\frac{dy}{dx} = 3xy$$

$$(4x + y)\,dx + xdy = 0$$

$$\frac{d^2u}{dx^2} + \frac{dv}{dx} = e^x$$

$$\frac{d^2y}{dx^2} - 9y = 0$$

are ordinary differential equations. On the other hand, if an equation involves partial derivatives of one or more dependent variables with respect to two or more independent variables, it is then called a **partial differential equation.**

Examples of such equations are

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 8u$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial u}{\partial x} = 5\frac{\partial v}{\partial y}$$

**NOTE** (i) Differential equations may have the variables present only in the derivatives. e.g., $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 = 0$

(ii) From the class of differential equations we shall exclude those equations which are identities such as $\frac{d}{dx}(\sin x) = \cos x$ or $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$.

### 7.1.3 DEFINITION

The **order** of a differential equation is the order of the highest derivative appearing in it. For example,

$$\frac{d^4y}{dx^4} - 2\left(\frac{dy}{dx}\right)^5 + 3y = 4x$$

is an ordinary differential equation of order 4.

In the equation $\left(\frac{d^3y}{dx^3}\right)\left(\frac{d^2y}{dx^2}\right) + 6\frac{dy}{dx} = 0$ we find that the third and second derivatives of y with respect to x are multiplied together. Nevertheless the order of the equation is three.

The equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$ is a second-order partial differential equation.

Although partial differential equations have their own importance but at present we shall confine our attention to the study of ordinary differential equations. In general, an ordinary differential equation of order n is symbolically represented as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

### 7.1.4 DEFINITION

If a differential equation can be rationalized and cleared of fractions with regard to all derivatives present, the exponent of the highest-order derivative is called the **degree** of the differential equation.

For example, the differential equation

$$(y'')^{2/3} = 4 + y'$$

can be rationalized by cubing both sides to obtain

$$(y'')^2 = (4 + y')^3$$

The exponent of the highest-order derivative present (namely $y''$) is 2. Hence the differential equation is of degree two.

However, the equation $y''' = \sqrt{2x + 3y}$ is of degree one.

**REMARK** Note that a differential equation may not have a degree. For example, the degree of the equation $6x^2 \frac{d^3y}{dx^3} + (\sin x) \frac{d^2y}{dx^2} - \cos xy = 0$ is undefined because the unknown function y is an argument of the transcendental cosine function and therefore, can not be written as a polynomial in y and its derivatives. Similarly, the degree of the differential equations

$$y'' + (y')^2 = \log y'' \text{ and } \sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + 3x + 2 \text{ is not defined.}$$

### 7.1.5 DEFINITION

A differential equation is said to be **linear** if (1) the dependent variable (say y) and its derivatives occur to the first degree only, (2) the products of y and its derivatives are absent and (3) no transcendental functions of y or its derivatives are present.

For instance, the differential equations

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

and

$$\frac{dy}{dx} - x^2y = \cos x$$

are linear as any of conditions (1) – (3) are not violated.

### 7.1.6 DEFINITION

A differential equation that is not linear is called **non-linear**.

Examples of non-linear differential equations are :

(1) $\frac{d^3y}{dx^3} + 4y^2 = 0$ (power of $y \neq 1$)

(2) $\frac{d^2y}{dx^2} + 7y\frac{dy}{dx} + 12y = 0$ (the term $7y\frac{dy}{dx}$ involves the product of dependent variable and its derivative)

(3) $\frac{d^2y}{dx^2} + \sin xy = 0$ (involves transcendental function of the dependent variable)

(4) $5\left[\frac{dy}{dx}\right]^3 + 2\frac{d^2y}{dx^2} + 3y = 0$ $\left(\text{the degree of } \frac{dy}{dx} \text{ is not } 1\right)$

In general, nth-order ordinary differential equation describing a functional relationship between the independent variable x, the dependent variable y and its derivatives, is said to be linear if it has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x) \quad \ldots\ldots \text{(i)}$$

where the coefficients $a_0(x), \cdots, a_n(x)$ and $f(x)$ are independent of y and $a_n(x) \neq 0$.

### 7.1.7 DEFINITION

Let us assume that in (i), $f(x) = 0$. Then it is known as a **homogeneous** differential equation. It is called homogeneous because the independent variable x appears, not by itself, but in conjunction with y or its derivatives. If $f(x)$ is not equal to zero then (i) is said to be an **in–homogeneous** differential equation. For example, the differential equation

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

is homogeneous and $m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F_1 \cos \omega t$ is non-homogeneous because cos wt appears independently of x or its derivatives.

## EXERCISE 7.1

State the order and the degree of each of the following differential equations

1. $\frac{dy}{dx} = 4y - x^2$

2. $(x + 2y)\,dx - (3x - y)\,dy = 0$

3. $5\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - y = e^{2x}$

4. $\left(\frac{dy}{dx}\right)^3 = \frac{2x}{3y}$

5. $\left(\frac{dy}{dx}\right)^3 = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

6. $\sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

7. $\frac{d^3y}{dx^3} = \sqrt[5]{\frac{dy}{dx}}$

8. $\frac{d^2x}{dy^2} - 4x = \cos y$

9. $\frac{d^2y}{dx^2} - 4xy = \sin\left(\frac{d^2y}{dx^2}\right)$

For each of the following state whether the equation is linear or non-linear :

10. $\frac{dy}{dx} + P(x)\, y = Q(x)$

11. $(x^2 - y^2)\, dx - 3xy\, dy = 0$

12. $\frac{d^2y}{dx^2} + 3y^2 = 2xy - 5$

13. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = \sin 3x$

14. $(2x - 3y)\, dx - (3x - 2y)\, dy = 0$

15. $\frac{dy}{dx} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$

## 7.2 DEFINITION

A function which is free of derivatives is called a **solution or integral** of the differential equation if the equation becomes an identity when the function and its derivatives are substituted in it.

For example $y = e^{-x}$ is a solution of $y'' - 2y' = 3y$, (where primes denote derivatives with respect to x) because substituting $e^{-x}$ for y, $-e^{-x}$ for $y'$ and $e^{-x}$ for $y''$ in $y'' - 2y' = 3y$, we get the identity

$$e^{-x} - 2(-e^{-x}) = 3e^{-x}$$

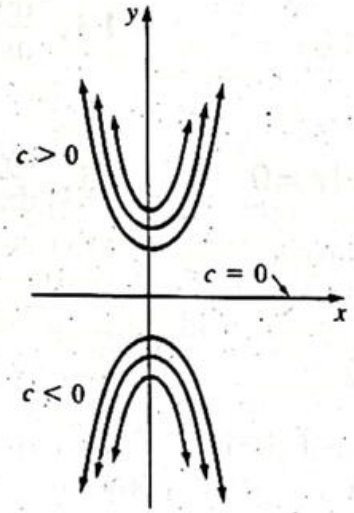
Similarly it can be easily seen that $y = e^{3x}$ is another solution of the same differential equation. A solution of a differential equation that is identically zero is often referred to as a trivial solution e.g. $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$. We notice that $y = 0$ is also a solution which is called the trivial solution.

It can easily be verified that $y=e^x$, $y=e^{-x}$, $y = Ae^x$, $y=Be^{-x}$ and $y = Ae^x+Be^{-x}$ are all solutions of the second-order differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

Moreover, it is obvious that $y = \frac{1}{2}x^2 + x$, $y = \frac{1}{2}x^2 + x + 8$ and $y = \frac{1}{2}x^2 + x - 24$ are solutions of the differential equation $\frac{dy}{dx} = x + 1$. Observe also that the general solution of the same differential equation is $y = \frac{1}{2}x^2 + x + c$, where c is an arbitrary constant. Since c can take any real value, this (general) solution can be represented graphically by an infinite set of curves.

Thus we observe that a differential equation usually possesses an infinite number of solutions. The one-parameter family $y = ce^{x^2}$, where c is an arbitrary constant, represents an infinite number of solutions of the differential equation $\frac{dy}{dx} = 2xy$. The trivial solution $y = 0$ is a member of this family of solutions corresponding to $c = 0$. (See the Figure)




**Note (1)** The graphs of the solutions of a differential equation are called its integral curves or solution curves.

**Note (2)** Not every differential equation has a solution.

For example, there cannot be a real function satisfying the differential equation

$$(y')^2 + 1 = 0$$

since, if there were, the left hand side of the equation would be greater than, or equal to one and thus could not possibly be zero. However, the equation has the non-real solutions $y = \pm ix$, where $i = \sqrt{-1}$

## 7.2.1 FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are formed not only out of consideration of families of geometric curves but also arise while describing physical problems in the sciences and engineering in mathematical terms/symbols.

Let us first consider a simple physical example which illustrates the typical steps leading from the physical situation to the mathematical formulation of a differential equation. Experiments suggest that a radio-active substance disintegrates at a rate proportional to the amount present. Let us denote the amount of substance still present at time t by Q(t). According to the law governing the process of radiation, the rate of change $\frac{dQ}{dt}$ is perportional to Q. This is expressed mathematically by the ordinary differential equation

$$\frac{dQ}{dt} = kQ$$

where k is a definite physical constant whose numerical value is known for different radioactive substances.

Newton conceived the law of gravitation and then solved the corresponding differential equations to show that the earth moves about the sun approximately in an ellipse with the sun at one focus. Today we use the same theory to learn about satellites, their orbits, and methods of guiding them.

Mathematically, given the equation of a family of curves, we can form the corresponding differential equation by differentiating and then eliminating the arbitrary constants.

Thus, to form the differential equation from $f(x , y, c_1, c_2, \dots , c_n) = 0$ we differentiate it with respect to x successively n-times and get

$$f_1(x, y, y', c_1, c_2, \dots , c_n) = 0$$

$$f_2(x, y, y', y'', c_1, c_2, \dots c_n) = 0$$

$$_ \quad _ \quad _ \quad _ \quad _ \quad _ \quad _ \quad _$$

$$_ \quad _ \quad _ \quad _ \quad _ \quad _ \quad _ \quad _$$

$$f_n(x, y, y', y'', \dots , y^n , c_1, c_2, \dots , c_n) = 0$$

Eliminating the n arbitrary constants $c_1, c_2, \dots , c_n$ from these (n + 1) equations we obtain

$$\phi (x, y, y', y'', \dots , y^n) = 0$$

which is the required nth – order differential equation.

Consider the following examples :

**EXAMPLE 1** Find the differential equation of the family of parabolas $y^2 = 4ax$.

**SOLUTION** We have $y^2 = 4ax$ ..... (i)

Differentiating (i) w.r.t. to x, we get

$$2y\frac{dy}{dx} = 4a = 4\left(\frac{y^2}{4x}\right)$$ [substituting for *a* from (i) ]

Since $y \neq 0$, the required differential equations is

$$\frac{dy}{dx} = \frac{y}{2x}$$

i.e., slope of the tangent at any point to (i) is the ratio of the ordinate to twice the obscissa of that point.

**EXAMPLE 2** Find the differential equation for $y = Ae^x + Be^{2x}$, where A and B are arbitrary constants.

**SOLUTION** $y = Ae^x + Be^{2x}$ ..... (i)

$$\therefore \frac{dy}{dx} = Ae^x + 2Be^{2x} \quad \text{..... (ii)}$$

and $$\frac{d^2y}{dx^2} = Ae^x + 4Be^{2x} \quad \text{..... (iii)}$$

Subtracting (i) from (ii) and (iii) respectively, we get

$$\frac{dy}{dx} - y = Be^{2x} \quad \text{..... (iv)}$$

and $$\frac{d^2y}{dx^2} - y = 3B\,e^{2x} \quad \text{..... (v)}$$

To eliminate B, substitute the value of $Be^{2x}$ form (iv) in (v) to get

$$\frac{d^2y}{dx^2} - y = 3\left(\frac{dy}{dx} - y\right)$$

Hence the required differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

**EXAMPLE 3** Form the differential equation by eliminating the arbitrary constants A, B and C, for $y^2 = Ax^2 + Bx + C$.

**SOLUTION** We have $y^2 = Ax^2 + Bx + C$

On differentiating w.r.t. x,

$$2y\frac{dy}{dx} = 2Ax + B$$

Differentiating again, we get

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A \quad \text{or} \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = A$$

Differentiating once again,

$$y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$$

which is a differential equation of order 3, obtained by eliminating three constants A, B, C.

**EXAMPLE 4** Find a differential equation for the family of all circles having radius 1 and centre anywhere in the xy–plane.

**SOLUTION** The equation of a circle with centre (h, k) and radius 1 is

$$(x-h)^2 + (y-k)^2 = 1 \qquad \ldots\ldots \text{(i)}$$

Differentiating with respect to x, we obtain

$$2(x-h) + 2(y-k)y' = 0$$

or $$x - h = -(y-k)y'$$

Substituting in (i) we have

$$(y-k)^2(y')^2 + (y-k)^2 = 1$$

so that the constant k is eliminated. To eliminate k, we solve for (y – k) to obtain

$$(y-k)^2 = \frac{1}{(y')^2+1} \quad \text{or} \quad y-k = \pm[1+(y')^2]^{-1/2}$$

Differentiating with respect to x, we get

$$y' = \pm\left[-\frac{1}{2}[1+(y')^2]^{-3/2}(2y'y'')\right]$$

or $$y' = \pm\frac{y'y''}{[1+(y')^2]^{3/2}}$$

yielding $$\frac{y''}{[1+(y')^2]^{3/2}} = \pm 1 \qquad \ldots\ldots \text{(ii)}$$

which is the required second order differential equation. Since the left hand side of (ii) represents the curvature of a plane curve, therefore, this equation states that the curvature of a certain plane curve at any point on it is equal to 1 in absolute value. Only circles of radius 1 have this property.

From example 1 we observe that the equation involving one arbitrary constant gives rise to differential equation of the first order. However, in example 2 it was necessary to introduce two derivatives to eliminate the two arbitrary constants. Thus we see that when we eliminate arbitrary constants from a given relation, the order of the resulting differential equation is the same as the number of arbitrary constants. This suggests that in general an equation containing n arbitrary constants should lead to an nth order differential equation.

## 7.2.2 GENERAL SOLUTION. PARTICULAR SOLUTION. SINGULAR SOLUTION.

### DEFINITIONS

The general solution (or complete primitive) of a differential equation is that in which the number of independent arbitrary constants is the same as the order of the differential equation.

By particular solution we mean the solution obtained by giving particular values to one or more of the independent arbitrary constants in the general solution

e.g., $y = mx + c$ is the general solution of the differential equation

$\frac{d^2y}{dx^2} = 0$ whereas $y = 3x + 5$ obtained by taking the particular values $m = 3$ and $c = 5$ is the particular solution.

A solution of a differential equation which cannot be obtained from the general solution by any choice of the independent arbitrary constants is called a singular solution e.g., the general solution of $y' = \sqrt{y}$ is $2\sqrt{y} = x + c$, and singular solution is $y = 0$.

**NOTE** The arbitrary constants appearing in the general solution of a differential equation must be independent and to test this we shall show that they cannot be replaced by or reduced to a smaller number of constants. Whenever we speak of arbitrary constants, we shall mean independent (or essential) arbitrary constants.

e.g., $y = l \sin(x + \alpha) + m \cos x$ being the solution of $\frac{d^2y}{dx^2} + y = 0$ seems to contain three constants $l$, m and $\alpha$. But they are not independent as they can be reduced to two only, for

$$y = l \sin x \cos\alpha + l \cos x \sin\alpha + m \cos x$$

$$= (l \cos\alpha) \sin x + (m + l \sin\alpha) \cos x$$

or $y = A \sin x + B \cos x.$

which contains only two arbitrary independent constants.

Similarly the relation $y = A\, e^{2x+B}$ has two constants A and B. But they are not independent (or essential) arbitrary constants, because

$y = A\, e^{2x+B} = A\, e^{B}\, e^{2x} = C\, e^{2x}$ where two constants A and B have been replaced essentially by one constant C.

## 7.2.3 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

In practice we are usually called upon to find the solution of a differential equation satisfying certain prescribed side conditions called initial or boundary conditions. These conditions help us in evaluating the arbitrary constants appearing in the general solution. If these conditions are specified at one point, such as

$$y(x_0) = a, y'(x_0) = a$$

then they are called initial conditions. A differential equation together with initial conditions is called initial value problem.

If the supplementary conditions on the solution of the differential equation are specified at more than one point they are known as boundary conditions. Typical boundary conditions consist of specifying the unknown function y, or its derivatives at two points, $x_0$ and $x_1$, as

$$\left.\begin{aligned} y(x_0) &= a \\ y(x_1) &= b \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} y(x_0) &= a \\ y'(x_1) &= b \end{aligned}\right\}$$

A differential equation together with boundary conditions is called boundary value problem.

**EXAMPLE 1** Solve the initial value problem

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12 = 0, \quad y(0) = -2, \quad y'(0) = 6$$

given that the differential equation has $y = c_1 e^{4x} + c_2 e^{-3x}$ as the general solution.

**SOLUTION** We have $y = c_1 e^{4x} + c_2 e^{-3x}$ ..... (i)

Since $y(0) = -2$, $-2 = c_1 + c_2$ ..... (ii)

Also $y' = 4c_1 e^{4x} - 3c_2 e^{-3x}$

$y'(0) = 6$ gives $6 = 4c_1 - 3c_2$ ..... (iii)

From (ii) and (iii) we obtain $c_1 = 0$, $c_2 = -2$.

Substituting these values in (i), we get

$$y = -2e^{-3x}$$

**EXAMPLE 2** Solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = 0 \ , \quad y(0) = 1, \ y'(\pi) = -1$$

where $y = A \sin x + B \cos x$

is the general solution of the given differential equation.

**SOLUTION** Using $y(0) = 1$ in $y = A \sin x + B \cos x$,

we get $1 = A \sin 0 + B \cos 0$ i.e., $B = 1$

Also $y' = A \cos x - B \sin x$

Since $y' = -1$ at $x = \pi$, so $-1 = A \cos \pi - B \sin \pi$,

or $-1 = -A$ i.e., $A = 1$

Thus the solution is

$$y = \sin x + \cos x.$$

## EXERCISE 7.2

Show that each function is a solution of the differential equation written opposite it:

| | |
|---|---|
| 1. $y = x^2 + cx$; | $xy' = x^2 + y$ |
| 2. $y = \frac{c}{x} + 1$; | $xy' + y = 1$ |
| 3. $y = 3 \tan 3x$; | $y' = 9 + y^2$ |
| 4. $y = e^{-x} + \sin x$; | $y'' + y = 2e^{-x}$ |
| 5. $y = Ae^x + Be^{-2x} + x^2 + x$; | $y'' + y' - 2y = 3 - 2x^2$ |
| 6. $y = e^{3x} \cos 2x$; | $y'' - 6y' + 13y = 0$ |
| 7. $y^3 - 3x + 3y = 5$ (y defined implicitly); | $y'' + 2y(y')^3 = 0$ |
| 8. $\log y + \frac{x}{y} = c$; | $(y - x)y' + y = 0$ |
| 9. $x = 4t + 1, y = t^2 - 2$; | $4(y')^2 = y + 2$ |

10. $u = e^{3x} \sin 2y$ ; $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = u$

11. $y = x^2 + x^2 \log x$ , $x > 0$ ; $x^2y'' - 3xy' + 4y = 0$

12. $y = A \cos x + \sin x$ $\cos x \frac{dy}{dx} + y \sin x = 1$

13. Find the values of the constants a and b if $y = ax + b$ is a solution of the differential equation $\frac{dy}{dx} = 4x - 2y$.

14. If $y = ax^2 + bx + c$ is a solution of the differential equation $\frac{d^2y}{dx^2} + x^2 = y$, find *a*, b and c.

Form the differential equation (by eliminating the arbitrary constant/s) whose general solution is

15. $y = mx + 3$ , m being arbitrary constant.

16. $y = (x + c)^2$ , family of parabolas.

17. $y = c(x - c)^2$

18. $x = A \cos(nt + \alpha)$ , where A, $\alpha$ are arbitrary constants.

19. $y = e^x(A \cos x + B \sin x)$ **20.** $y = ax + bx^3$

21. $y = c_1 e^{2x} + c_2 e^{-x} + x$ **22.** $2x^2 + 3y^2 + Ax + By + C = 0$

23. $Ax^2 + By^2 = 1$ **24.** $x^2 + y^2 - 2kx = 0$

25. $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$

Find the differential equation of

26. the family of circles (i) centred at the origin, (ii) having their centres on y-axis

27. the family of circles passing through the origin with centre on the y-axis

28. the family of circles passing through the origin

29. the family of circles of radius *a*

30. the family of parabolas having their vertices at the origin and their foci on the y-axis

31. all tangent lines to the parabola (i) $y = x^2$ (ii) $y^2 = 4x$

32. all parabolas of latus rectum $4a$ and axes parallel to x-axis

33. all parabolas having their axes parallel to y-axis

34. all conics whose axes coincide with the axes of coordinates.

35. the system of ellipses having their axes along the x-axis and the y-axis.

36. Solve the initial-value problem

$$y'' = \sin 2x \quad \text{and} \quad y(0) = 1,\ y'(0) = 2$$

if every solution of the given differential equation is defined by

$$y(x) = -\frac{1}{4}\sin 2x + c_1 + c_2 x.$$

37. Given that every solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$$

may be written in the form $y = c_1 e^{4x} + c_2 e^{-3x}$, for some choice of the arbitrary constants $c_1$ and $c_2$, solve the following initial value problems:

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$, $y(0) = 5$, $y'(0) = 6$.

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$, $y(0) = -2$, $y'(0) = -8$.

38. If every solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ may be written in the form $y = c_1 \sin x + c_2 \cos x$, for some choice of arbitrary constants $c_1$ and $c_2$, show that boundary problems (i), (ii) and (iii) possess solutions but (iv) and (v) do not.

(i) $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$.

(ii) $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 1$, $y'\left(\frac{\pi}{2}\right) = -1$.

(iii) $\frac{d^2y}{dx^2} + y = 0$, $y(1) = 3$, $y'(1) = -4$.

(iv) $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 1$, $y(\pi) = 5$.

(v) $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 0$, $y(\pi) = 1$.

## 7.3 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

We shall now consider some special types of equations of first order and first degree and the corresponding methods used for derivation of their solution. It will be assumed that the conditions for the existence of solution are satisfied. An ordinary differential equation of first order and first degree $f\left(x, y, \frac{dy}{dx}\right) = 0$ can be expressed in either the form $\frac{dy}{dx} = f(x, y)$ or the differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. The general solution of such an equation will contain only one arbitrary constant. It may be remarked that it is not always possible to solve such equations. However, we shall discuss some special standard methods of solution which can be applied to equations classified as follows.

(i) Equations with separable variables.

(ii) Homogeneous equations.

(iii) Exact equations.

(iv) Linear equations.

### 7.3.1 EQUATIONS WITH SEPARABLE VARIABLES

If an equation of the first order and first degree can be written in the form

$$M(x)\,dx + N(y)\,dy = 0, \qquad \ldots\ldots \text{(i)}$$

where $M(x)$ is a function of x alone and $N(y)$ is a function of y alone, we call it a separable equation and solve it by integrating each term separately. Thus the solution of equation (i) will be

$$\int M(x)\,dx + \int N(y)\,dy = c, \text{ where c is constant of integration.}$$

On solving this equation for y in terms of x, we have

$$y = y(x, c)$$

Note that the constant of integration c can be replaced by $\log c$, $e^c$, $\sin c$, $\tan^{-1} c$ etc. whichever is convenient or suitable for simplification.

**EXAMPLE 1** Solve $\frac{dy}{dx} = 2x(1 + y^2)$

**SOLUTION** Separating the variables, we obtain

$$\frac{dy}{1 + y^2} = 2x\,dx$$

Integrating both sides

$$\tan^{-1}y = x^2 + c$$

Solving for y, we get

$$y = \tan(x^2 + c)$$

**EXAMPLE 2** Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

**SOLUTION** From the given equation,

$$\frac{dy}{dx} = e^{3x} e^{-2y} + x^2 e^{-2y}$$

$$= e^{-2y}(e^{3x} + x^2)$$

Separating the variables,

$$e^{2y}\,dy = (e^{3x} + x^2)\,dx$$

Integration yields,

$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c_1$$

or $3e^{2y} = 2(e^{3x} + x^3) + c$, where $c = 6c_1$.

**EXAMPLE 3** Solve

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

**SOLUTION** The given equation can be written as

$$y - ay^2 = (x + a)\frac{dy}{dx}$$

Separating the variables, we have

$$\frac{dy}{y(1 - ay)} = \frac{dx}{x + a}$$

$$\left(\frac{1}{y} + \frac{a}{1 - ay}\right)dy = \frac{1}{x + a}\,dx$$

Integrating both sides,

$$\log y^* - \log(1 - ay) = \log(x + a) + c_1$$

or $$\log \frac{y}{1 - ay} = \log c(x + a) \quad , \quad \text{where } c_1 = \log c$$

or $$\frac{y}{1 - ay} = c(x + a)$$

Hence $$y = c(1 - ay)(x + a)$$

or $$y = c(x + a) - ca(x + a)y$$

or $$y(1 + cax + ca^2) = c(x + a)$$

or $$y = \frac{c(x + a)}{1 + cax + ca^2}$$ is the required solution.

**EXAMPLE 4** Solve the initial value problem

$$8\cos^2 y\, dx + \operatorname{cosec}^2 x\, dy = 0 \quad , \quad y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$$

**SOLUTION** Separating the variables, we get

$$\frac{8}{\operatorname{cosec}^2 x} dx + \frac{1}{\cos^2 y} dy = 0$$

or $$8\sin^2 x\, dx + \sec^2 y\, dy = 0$$

or $$8\left[\frac{1 - \cos 2x}{2}\right] dx + \sec^2 y\, dy = 0$$

or $$4[1 - \cos 2x]\, dx + \sec^2 y\, dy = 0$$

Integrating, we obtain

$$4x - 2\sin 2x + \tan y = c$$

Applying the initial condition $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$, we get

---

* We shall use $\int \frac{du}{u} = \log u$, with the understanding that $u > 0$. Actually to take care of the cases where $u > 0$ or $u < 0$, one should prefer the use of $\int \frac{du}{u} = \log |u|$. When no base is specified the base 'e' is understood.

$$\frac{\pi}{3} - 2\left(\frac{1}{2}\right) + 1 = c \quad \text{or} \quad c = \frac{\pi}{3}$$

Hence the required particular soltuion is

$$4x - 2\sin 2x + \tan y = \frac{\pi}{3}$$

$$\text{i.e.,} \quad \tan y = 2\sin 2x - 4x + \frac{\pi}{3}$$

## 7.3.2 USE OF SUBSTIUTION

An equation of the type

$$\frac{dy}{dx} = f(ax + by + c) \text{ or } \frac{dy}{dx} = f(ax + by), \quad b \neq 0,$$

which is not expressible in the separable form, can be transformed to an equation in which variables can be separated by means of the substitution $u = ax + by + c$ or $u = ax + by$ as procedured below.

**EXAMPLE 1** Solve $(x - y)^2 \frac{dy}{dx} = a^2$

**SOLUTION** Here we suppose that $u = x - y$ so that on differentiating, we get

$$\frac{du}{dx} = 1 - \frac{dy}{dx} \qquad \text{i.e.,} \qquad \frac{dy}{dx} = 1 - \frac{du}{dx}$$

$\therefore$ given equation becomes

$$u^2\left(1 - \frac{du}{dx}\right) = a^2 \quad \text{or} \quad u^2 - a^2 = u^2\frac{du}{dx}$$

Now separating the variables,

$$dx = \left(\frac{u^2}{u^2 - a^2}\right)du = \left(\frac{u^2 - a^2 + a^2}{u^2 - a^2}\right)du = \left(1 + \frac{a^2}{u^2 - a^2}\right)du$$

Integrating, $x + c = u + a^2 \cdot \frac{1}{2a} \log \frac{u - a}{u + a}$

Changing the variables back again by putting $u = x - y$, we get

$$x + c = x - y + \frac{a}{2} \log \frac{x - y - a}{x - y + a}$$

$$\text{or } 2y + 2c = a \log \frac{x - y - a}{x - y + a}$$

$$\text{or } 2y + k = a \log \frac{x - y - a}{x - y + a} \text{ as the required solution.}$$

**EXAMPLE 2** Solve

$$\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$$

**SOLUTION** Let $u = x + y$

Differentiating $\frac{du}{dx} = 1 + \frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{du}{dx} - 1$

The given equation then becomes

$$\frac{du}{dx} - 1 = \sin u + \cos u \text{ or } \frac{du}{dx} = 1 + \sin u + \cos u$$

$$\text{or } \frac{du}{dx} = 1 + 2\sin \frac{u}{2} \cos \frac{u}{2} + 2\cos^2 \frac{u}{2} - 1$$

$$= 2\cos^2 \frac{u}{2} \left[1 + \tan \frac{u}{2}\right]$$

Separating the variables, we get

$$dx = \frac{du}{2\cos^2 \frac{u}{2} \left(1 + \tan \frac{u}{2}\right)} = \frac{\frac{1}{2}\sec^2 \frac{u}{2}}{1 + \tan \frac{u}{2}} du$$

Integration gives

$$x + c = \log \left(1 + \tan \frac{u}{2}\right)$$

$$= \log \left(1 + \tan \frac{x + y}{2}\right), \text{ substituting } u = x + y$$

**EXAMPLE 3** Solve the differential equation

$$e^{-y}\left(\frac{dy}{dx} + 1\right) = xe^{x}$$

**SOLUTION** We have $\frac{dy}{dx} + 1 = xe^{x}e^{y} = xe^{x+y}$ ..... (i)

Making the substitution $u = x + y$ so that

$\frac{du}{dx} = 1 + \frac{dy}{dx}$, we obtain from (i)

$\frac{du}{dx} = xe^{u}$ ; $\frac{du}{e^{u}} = xdx$

Integrating $-e^{-u} = \frac{x^2}{2} + c_1$

or $e^{-u} + \frac{x^2}{2} = c$ , where $c = -c_1$

or $e^{-(x+y)} + \frac{x^2}{2} = c.$

Taking logarithms, $-(x + y)\log e = \log\left(c - \frac{x^2}{2}\right)$

or $y = -\log\left(c - \frac{x^2}{2}\right) - x.$

## EXERCISE 7.3

Find the general solution of each differential equation and, when initial conditions are given, the corresponding particular solution:

1. $(a - x)\,dy - (a + y)\,dx = 0$ ,

2. $(x + 1)\frac{dy}{dx} = x(y^2 + 1)$,

3. $(\sin x + \cos x)\,dy + (\cos x - \sin x)\,dx = 0$

4. $x\frac{dy}{dx} = y/\log x$

5. $(1 + 2y^2)\,dy = y\cos x\,dx$, $y(0) = 1$

6. $\sqrt{1 - y^2}\,dx = \sqrt{1 - x^2}\,dy$ ; $y = \frac{1}{2}$ when $x = 1$

7. $2x(y+1)\,dx - y\,dy = 0$ ; $y = -2$ when $x = 0$

8. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

9. $\frac{dy}{dx} + x^2 = x^2 e^{3y}$

10. $[\cos x \tan y + \cos(x+y)]\,dx + [\sin x \sec^2 y + \cos(x+y)]dy = 0$

11. $\cos(x+y)\,dy = dx$

12. $(x^2 - ay)\,dx = (ax - y^2)\,dy$

13. $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

14. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

15. $x \sin x\, e^{-y}\,dx - y\,dy = 0$

16. $(y + x^2 y)\frac{dy}{dx} = 2x$, $y(0) = -2$

17. $x\frac{dy}{dx} + \cot y = 0$ ; $y = \frac{\pi}{4}$ , $x = \sqrt{2}$

18. $(x+y+1)^2 \frac{dy}{dx} = 1$

19. $\frac{dy}{dx} = (4x + y + 1)^2$

20. $y - x\frac{dy}{dx} = a\left(1 + x^2 \frac{dy}{dx}\right)$

21. $x\sin y\,dx + (x^2+1)\cos y\,dy = 0$ , $y(1) = \frac{\pi}{2}$

22. $x\cos x\,dx + (1 - 6y^5)\,dy = 0$ , $y(\pi) = 0$

## 7.4 HOMOGENEOUS DIFFERENTIAL EQUATIONS

### DEFINITION

If $f(tx, ty) = t^n f(x, y)$ for some real number n, then the function f(x, y) is called a **homogeneous function** of degree n. This means that if tx and ty are substituted for x and y respectively in f(x, y) and if $t^n$ is factored out, the other factor that remains is the original expression f(x, y) itself. For example, consider the function

(i) $f(x, y) = 2x^3 - xy^2$

so that $f(tx, ty) = 2(tx)^3 - (tx)(ty)^2$

$= t^3[2x^3 - xy^2] = t^3 f(x, y)$

Thus f(x, y) is homogeneous function of degree 3.

(ii) $f(x, y) = x^2 - y^2 + 3$

$$f(tx, ty) = t^2x^2 - t^2y^2 + 3 \neq t^2 f(x, y),$$

since $t^2f(x, y) = t^2x^2 - t^2y^2 + 3t^2.$

Hence in this case the function f(x, y) is not homogeneous.

(iii) $f(x, y) = \frac{2y}{3x} + 7$

$$f(tx, ty) = \frac{2ty}{3tx} + 7 = \frac{2y}{3x} + 7 = t^0 f(x, y)$$

Thus the function f(x, y) is homogeneous of degree zero. In fact, any function of $\frac{y}{x}$ is homogeneous of degree zero since $f\left(\frac{ty}{tx}\right) = t^0 f\left(\frac{y}{x}\right)$.

It may also be noted that in many cases a homogeneous funetion can be recognized by examining the total degree of each term. Consider

$$f(x, y) = 3x^3y - 2x^2y^2$$

Here the degree of the term $3x^3y = 3 + 1 = 4$ and degree of the term

$2x^2y^2 = 2 + 2 = 4$. Hence the function f(x, y) is homogeneous of degree 4.

A useful relation about homogeneity of f(x, y) is obtained by substituting $t = \frac{1}{x}$ in the definition $f(tx, ty) = t^n f(x, y)$. This leads to

$$\left(\frac{1}{x}\right)^n f(x, y) = f\left(\frac{x}{x}, \frac{y}{x}\right) = \varphi\left(\frac{y}{x}\right)$$

so that

$$f(x, y) = x^n f\left(1, \frac{y}{x}\right) = x^n \varphi\left(\frac{y}{x}\right)$$

### 7.4.1 DEFINITION

A differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be **homogeneous** in x and y if M and N are homogeneous functions of the same degree in x and y.

Since we can write $M(x, y)\,dx + N(x, y)\,dy = 0$

in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = -\frac{x^n M\left(1, \frac{y}{x}\right)}{x^n N\left(1, \frac{y}{x}\right)} = -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} = g\left(\frac{y}{x}\right)$$

hence we can give the following alternate equivalent definition as well.

### 7.4.2 DEFINITION

A differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is called homogeneous if, when expressed in the form $\frac{dy}{dx} = f(x, y)$, there exists a function 'g' such that $f(x, y)$ can be written in the form $g\left(\frac{y}{x}\right)$.

The important role played by $\frac{y}{x}$ in a homogeneous function suggests that the substitution $\frac{y}{x} = v$ or $y = vx$ might be effective in solving a homogeneous differential equation of the first order and first degree. We shall now prove that this substitution reduces a homogeneous equation to an equation of the type 'variables separable'.

### 7.4.3 THEOREM

The substitution $\frac{y}{x} = v$ or $y = vx$ transforms the homogeneous differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad \ldots\ldots \text{(i)}$$

into an equation in v (new dependent variable) and x in which the variables are separable.

**PROOF** Since equation (i) is homogeneous, it can be put in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \qquad \ldots\ldots \text{(ii)}$$

Let us introduce a new dependent variable v by making the substitution

$\frac{y}{x} = v$ i.e., $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Hence equation (ii) converts to

$$v + x\frac{dv}{dx} = g(v)$$

or $$[v - g(v)] + x\frac{dv}{dx} = 0$$

which is a separable equation and can be solved thus. Separation of variables and integration gives

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c \text{ , where c is a constant of integration.}$$

$$\text{or } f(v) + \log x = c \text{ , where } f(v) = \int \frac{dv}{v - g(v)}.$$

But $v = \frac{y}{x}$, therefore

$$f\left(\frac{y}{x}\right) + \log x = c$$

which is the required solution.

**NOTE** In place of $y = vx$ we can use the substitution $x = vy$ as well. In solving equation (i) the transformation

$y = vx$ is sometimes more useful when N is simpler than M and $x = vy$ is of advantage when M is relatively simpler than N.

**EXAMPLE 1** Solve $x^2\left(\frac{dy}{dx} - 1\right) = y(x + y)$.

**SOLUTION** We can write the given equation as

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2 \quad \text{(homogeneous)}$$

$$\text{or } \frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2} \qquad \ldots\ldots \text{(iii)}$$

Let $v = \frac{y}{x}$ or $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then the equation (iii) becomes

$$v + x\frac{dv}{dx} = 1 + v + v^2$$

$$\text{or } x\frac{dv}{dx} = 1 + v^2$$

Separating the variables,

$$\frac{dv}{1 + v^2} = \frac{dx}{x}$$

Integrating we get

$$\tan^{-1} v = \log x + c_1$$

But $v = \frac{y}{x}$, therefore

$$\tan^{-1}\left(\frac{y}{x}\right) = \log x + \log c \text{ , where we take } c_1 = \log c,\ c > 0 \text{ for the sake of simplification}$$
$$= \log cx$$

or $$\frac{y}{x} = \tan(\log cx)$$

Hence $y = x \tan(\log cx)$ is the required solution.

**EXAMPLE 2** Solve

$$\left(x \tan\frac{y}{x} - y \sec^2\frac{y}{x}\right)dx + x \sec^2\frac{y}{x}\,dy = 0$$

**SOLUTION** We can write

$$\tan\left(\frac{y}{x}\right) - \frac{y}{x}\sec^2\left(\frac{y}{x}\right) + \sec^2\left(\frac{y}{x}\right)\frac{dy}{dx} = 0 \quad \text{(homogeneous)}$$

Let $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then the above equation becomes

$$\tan v - v \sec^2 v + \sec^2 v\left[v + x\frac{dv}{dx}\right] = 0$$

or $$\tan v + x \sec^2 v \frac{dv}{dx} = 0$$

Separating the variables, we obtain

$$\frac{\sec^2 v}{\tan v}\,dv + \frac{dx}{x} = 0$$

Integrating

$$\log \tan v + \log x = \log c$$

or $$x \tan v = c$$

or $$x \tan \frac{y}{x} = c$$

or $$y = x \tan^{-1}\left(\frac{c}{x}\right)$$

**EXAMPLE 3** Solve

$$y(ydx - xdy) + x\sqrt{x^2 + y^2}\ dy = 0$$

**SOLUTION** The given equation can be rearranged as

$$\frac{dy}{dx} = \frac{y^2}{xy - x\sqrt{x^2 + y^2}} = \frac{y^2/x^2}{y/x - \sqrt{1 + y^2/x^2}} = \varphi\left(\frac{y}{x}\right) \text{ (Homogeneous)}$$

Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ it becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2\sqrt{1 + v^2}}$$

or $$x\frac{dv}{dx} = \frac{v^2}{v - \sqrt{1 + v^2}} - v = \frac{v^2 - v^2 + v\sqrt{1 + v^2}}{v - \sqrt{1 + v^2}}$$

Separating the variables,

$$\left(\frac{v - \sqrt{1 + v^2}}{v\sqrt{1 + v^2}}\right) dv = \frac{dx}{x}$$

or $$\left(\frac{1}{\sqrt{1 + v^2}} - \frac{1}{v}\right) dv = \frac{dx}{x}$$

Integrating, $\log\left[v + \sqrt{1 + v^2}\right] - \log v = \log x + \log c$

or $$\log\left(\frac{v + \sqrt{1 + v^2}}{v}\right) = \log cx$$

or $$\frac{v + \sqrt{1 + v^2}}{v} = cx$$

But $v = \frac{y}{x}$. Therefore

$$y + \sqrt{x^2 + y^2} = cxy$$

**EXAMPLE 4** Solve the initial value problem

$$x\frac{dy}{dx} - y = x\cos^2\left(\frac{y}{x}\right); \quad y(1) = \frac{\pi}{4}$$

**SOLUTION** We can write the given equation as

$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right) \qquad \left(\text{Homogeneous; } g\left(\frac{y}{x}\right)\right)$$

Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = v + \cos^2 v$$

or $$\sec^2 v\, dv = \frac{dx}{x}$$

Integrating, $$\tan v = \log x + c$$

But $y = vx$, therefore

$$\tan\left(\frac{y}{x}\right) = \log x + c$$

Applying $y(1) = \frac{\pi}{4}$, we get $\tan\frac{\pi}{4} = \log 1 + c \Rightarrow c = 1$

Hence the required solution is

$$\tan\frac{y}{x} = \log x + 1$$

## EXERCISE 7.4

Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$
2. $x\frac{dy}{dx} = y + x\,e^{y/x}$
3. $(x + y)dy = (x - y)\,dx$
4. $(x + y)\,dx - (x - y)\,dy = 0$
5. $(x^2 + y^2)\,dx = 2xy\,dy$
6. $(x^2 - y^2)dx = 2xydy$
7. $(x^2 + y^2)dx + x(x - y)\,dy = 0$
8. $x^2\,dy + y(x + y)\,dx = 0$

9. $\left(2\sqrt{xy} - y\right) dx - x dy = 0$

10. $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

11. $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

12. $(x^2 - xy + y^2)\,dx - xy dy = 0$

13. $xy\,dx + (x^2 + y^2)\,dy = 0$

14. $y^2 - x(y + x)\frac{dy}{dx} = 0$

15. $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)\,dy = 0$

16. $(xy^2 + 2x^3)\,dy = y^3 dx$

17. $(x^2 + 3xy + y^2)dx - x^2 dy = 0$

18. $xy^2 dy = (y^3 - 2x^3)dx$

19. $x \sin\left(\frac{y}{x}\right)dy = \left[y \sin\left(\frac{y}{x}\right) \quad x\right]dx$

20. $x \cos\left(\frac{y}{x}\right)[y\,dx + x\,dy] = y \sin\left(\frac{y}{x}\right)[x\,dy - y\,dx]$

Solve the initial value problems

21. $x\,dy = (x + y)\,dx\,;\ y(1) = -1$

22. $xy\,dy = (x^2 + y^2)\,dx;\ y(1) = -2$

23. $xdy - ydx = \sqrt{x^2 + y^2}\,dx;\ y(1) = 0$

24. $(x^2 + 3y^2)dx = 2xy\,dy;\ y(2) = 6.$

## 7.5 EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM

Consider the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \quad \text{where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \qquad \ldots\ldots \text{(i)}$$

The above equation is not homogeneous because of the presence of constants $c_1$ and $c_2$. In order to reduce it to the homogeneous form, we remove $c_1$ and $c_2$ by making the transformation

$$x = X + h, \quad y = Y + k \text{ so that } \frac{dy}{dx} = \frac{dY}{dX}$$

where X and Y are new variables and h and k are constants to be so chosen that the resulting equation in terms of X and Y becomes homogeneous.

$$\therefore \frac{dY}{dX} = \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2}$$

$$= \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \quad \ldots\ldots \text{(ii)}$$

Choose h and k such that

$$a_1h + b_1k + c_1 = 0,$$

and $a_2h + b_2k + c_2 = 0.$

Solving these two equations, we get

$$\frac{h}{b_1c_2 - b_2c_1} = \frac{k}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\text{or } h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad k = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad \ldots\ldots \text{(iii)}$$

Given that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, therefore $a_1b_2 - a_2b_1 \neq 0$. Hence h and k will exist and substitution of their values in the equation (ii) will lead to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous in the variables X and Y and can be solved by putting $\frac{Y}{X} = v$ or $Y = vX$ as discussed earlier. The solution will be in terms of X and Y which can be converted back to the original variables x and y by using $X = x - h$ and $Y = y - k$ respectively where h, k are given by (iii).

### 7.5.1 CASE OF FAILURE

If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then the above method fails because $a_1b_2 - a_2b_1 = 0$ and hence the values of h and k given by (iii) are indeterminate.

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{k}$, say.

Therefore $a_2 = ka_1$, $b_2 = kb_1$ and the given differential equation (i) becomes

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2} \quad \ldots\ldots \text{(iv)}$$

Making the substitution $a_1x + b_1y = v$, we obtain

$$\frac{1}{b_1}\left(\frac{dv}{dx} - a_1\right) = \frac{v + c_1}{kv + c_2}$$

or $$\frac{dv}{dx} = a_1 + b_1\left(\frac{v + c_1}{kv + c_2}\right)$$

in which the variables are separable and can be solved by the usual method in terms of v and x. Replacing v by $a_1x + b_1y$ we get the required solution.

**EXAMPLE 1** Solve

$$(x - y - 1)\,dy = (x + 3y - 5)\,dx$$

**SOLUTION** We can write

$$\frac{dy}{dx} = \frac{x + 3y - 5}{x - y - 1} \qquad \ldots\ldots \text{(i)}$$

Note that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Let $x = X + h$, $y = Y + k$ so that $\frac{dy}{dx} = \frac{dY}{dX}$.

Then equation (i) becomes

$$\frac{dY}{dX} = \frac{X + h + 3(Y + k) - 5}{X + h - (Y + k) - 1} = \frac{X + 3Y + (h + 3k - 5)}{X - Y + (h - k - 1)}$$

We choose h, k so that $h + 3k - 5 = 0$,

and $h - k - 1 = 0$

Solving there two equations we get $h = 2, k = 1$.

$$\therefore \frac{dY}{dX} = \frac{X + 3Y}{X - Y}$$ (Homogeneous)

Putting $Y = vX$ so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$, we obtain

$$v + X\frac{dv}{dX} = \frac{X + 3vX}{X - vX} = \frac{1 + 3v}{1 - v}$$

$$\therefore X\frac{dv}{dX} = \frac{1 + 3v}{1 - v} - v = \frac{(1 + v)^2}{1 - v}$$

Separating the variables

$$\frac{1 - v}{(1 + v)^2}\, dv = \frac{dX}{X}$$

$$\left[\frac{1}{(1 + v)^2} - \frac{1 + v - 1}{(1 + v)^2}\right] dv = \frac{dX}{X}$$

$$\left[\frac{2}{(1 + v)^2} - \frac{1}{1 + v}\right] dv = \frac{dX}{X}$$

Integrating, we get

$$-\frac{2}{1 + v} - \log (1 + v) = \log X + \log c$$

$$-\frac{2}{1 + \frac{Y}{X}} - \log\left(1 + \frac{Y}{X}\right) = \log X + \log c$$

$$-\frac{2X}{X + Y} - \log (X + Y) + \log X = \log X + \log c$$

$$-\frac{2X}{X + Y} = \log c\,(X + Y)$$

Converting X and Y back to original variables x and y, we get

$$-\frac{2(x - 2)}{(x - 2) + (y - 1)} = \log c\,(x - 2 + y - 1)$$

$$4 - 2x = (x + y - 3) \log c\,(x + y - 3)$$

**EXAMPLE 2** Solve

$$\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}$$

**SOLUTION** Since $\frac{a_1}{a_2} = 1 = \frac{b_1}{b_2}$, so we put

$$3x - 4y = v. \text{ Then } 3 - 4\frac{dy}{dx} = \frac{dv}{dx}$$

or $$\frac{dy}{dx} = -\frac{1}{4}\left(\frac{dv}{dx} - 3\right).$$

Therefore, the given equation becomes

$$-\frac{1}{4}\left(\frac{dv}{dx} - 3\right) = \frac{v - 2}{v - 3}$$

or $$\frac{dv}{dx} = \frac{-4v + 8}{v - 3} + 3 = -\frac{v + 1}{v - 3}$$

Separating the variables

$$\frac{v - 3}{v + 1}\,dv = -dx$$

$$\left(1 - \frac{4}{v + 1}\right)dv = -dx$$

Integrating

$$v - 4\log(v + 1) = -x + c_1$$

$$(3x - 4y) - 4\log(3x - 4y + 1) = -x + c_1$$

$$4x - 4y - 4\log(3x - 4y + 1) = c_1$$

$$x - y - \log(3x - 4y + 1) = c, \text{ where } c = \frac{c_1}{4}$$

## EXERCISE 7.5

Solve the following differential equations

1. $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

2. $\frac{dy}{dx} = \frac{2x - 6y + 7}{x - 3y + 4}$

3. $\frac{dy}{dx} = \frac{2y - x + 5}{2x - y - 4}$

4. $(2x+y+1)dx + (4x+2y-1)dy = 0$

5. $(2x-3y+4)dx + (3x-2y+1)dy = 0$

6. $(x + 2y)(dx - dy) = dx + dy$

7. $(2x + 3y - 1)dx + (2x + 3y + 2)dy = 0$, subject to the initial conditions $y = 3$ when $x = 1$.

## 7.6 EXACT DIFFERENTIAL EQUATIONS

**7.6.1 DEFINITION** An equation

$$M(x,y)\,dx + N(x,y)dy = 0$$

obtained by equating an exact differential to zero, without any operation of reduction or elimination, is known as an **exact differential equation.**

For example, $2xy\,dx + x^2\,dy = 0$ is an exact differential equation since the left hand side is the exact differential of $x^2y$. But the equation $2y\,dx + x\,dy = 0$ is not

## 7.6.2 CONDITION FOR EXACTNESS

In order to check whether or not a given differential equation is exact, we give the following theorem.

## 7.6.3 THEOREM

The necessary and sufficient condition for the ordinary differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad \ldots\ldots \text{(i)}$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where M and N possess continuous first order partial derivatives.

### PROOF. NECESSARY CONDITION

Let equation (i) be exact, then M dx + N dy is an exact differential of some function of x and y, say f(x,y). Hence

$$df = M\,dx + N\,dy$$

But $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$\therefore \quad M = \frac{\partial f}{\partial x} \text{ and } N = \frac{\partial f}{\partial y}$$

Thus $$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But M and N possess continuous first order partial derivatives, therefore

$$\frac{\partial^2 x}{\partial y \partial x} = \frac{\partial^2 x}{\partial x \partial y}$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which is the necessary condition for the given equation to be exact.

### SUFFICIENT CONDITION

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Suppose that $\int M \partial x = v$ is partial integral of M w.r. t. x treating y as a constant.

Then $$\frac{\partial v}{\partial x} = M$$

$$\therefore \quad \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating both sides of $\frac{\partial N}{\partial x} = \frac{\partial^2 v}{\partial y \partial x}$ we have

$$N = \frac{\partial v}{\partial y} + \phi(y)$$

where $\phi(y)$ is some function of y alone.

$$\therefore \quad Mdx + Ndy = \frac{\partial v}{\partial x} dx + \left[\frac{\partial v}{\partial y} + \phi(y)\right] dy$$

$$= \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) + \phi(y)\, dy$$

$$= dv + \phi(y)\, dy$$

$$= dv + d\left(\int \phi(y)\, dy\right)$$

$$= d\left[(v + \int \phi(y)\, dy\right]$$

$\therefore$ $Mdx + Ndy$ is an exact differential.

$\therefore$ the differential equation $Mdx + Ndy = 0$ is exact.

### 7.6.4 METHOD OF SOLVING an exact differential equation

$$Mdx + Ndy = 0.$$

While proving the sufficiency we have seen that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow Mdx + Ndy = d\left[v + \int \phi(y)\, dy\right]$$

$$\therefore \quad Mdx + Ndy = 0 \Rightarrow d\left[v + \int \phi(y)\, dy\right] = 0$$

Integrating, we obtain

$$v + \int \phi(y)\, dy = c$$

or

$$\int Mdx + \int \phi(y)\, dy = c$$

which gives the solution of the exact equation (i). It may be noted that $\int Mdx$ stands for the integral of M w.r. to x keeping y as constant and $\phi(y)$ stands for those terms of N that do not contain x.

Hence we have the following working rule for the solution of exact differential equation (i).

(1) Integrate M w.r. t. x treating y as a constant.

(2) Integrate w.r. t. y, those terms in N which do not involve x.

(3) Equate the sum of these integrals to a constant to obtain the required solution.

In symbols, $\int_{y \text{ constant}} Mdx + \int (\text{terms of N not containing x})dy = c$

**EXAMPLE 1** Solve the initial value problem

$(y + 2xy^2)dx + (x + 2x^2y)dy = 0 \; ; \; y = 1 \text{ when } x = 1$

**SOLUTION** Here $M = y + 2xy^2$ so that $\frac{\partial M}{\partial y} = 1 + 4xy$

and $N = x + 2x^2y$ so that $\frac{\partial N}{\partial x} = 1 + 4xy$

Since $\frac{\partial M}{\partial y} = 1 + 4xy = \frac{\partial N}{\partial x}$, therefore the given equation is exact and its solution is given by

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of N not containing x})\, dy = c$$

or $$yx + 2\left(\frac{x^2}{2}\right)y^2 = c$$

But $y = 1$ when $x = 1$. Therefore $c = 2$

Hence the required solution is $yx + x^2y^2 = 2$

**EXAMPLE 2** Determine if the equation

$$(y \cos x + 2xe^y)\, dx + (\sin x + x^2e^y - 1)\, dy = 0$$

is exact. If so, solve it.

**SOLUTION** We have $M = y\cos x + 2xe^y$, $N = \sin x + x^2e^y - 1$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + 2xe^y \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x + 2xe^y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; therefore, the given equation is exact.

Hence the solution is given by

$$\int_{y \text{ constant}} (y\cos x + 2xe^y)\,dx + \int (-1)dy = c$$

or $$y\sin x + x^2e^y - y = c$$

**EXAMPLE 3** Solve

$$(x\,dx + y\,dy)(x^2 + y^2) = a^2(x\,dy - y\,dx)$$

**SOLUTION** The given equation can be written as

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2 + y^2}$$

Rearranging into the standard form

$$\left(x + \frac{a^2y}{x^2 + y^2}\right)dx + \left(y - \frac{a^2x}{x^2 + y^2}\right)dy = 0$$

Here $M = x + \dfrac{a^2y}{x^2 + y^2}$ and $N = y - \dfrac{a^2x}{x^2 + y^2}$

so $$\frac{\partial M}{\partial y} = a^2\left[\frac{1\cdot(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

and $$\frac{\partial N}{\partial x} = -a^2\left[\frac{1\cdot(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}\right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ implies that the given equation is exact and its solution is

$$\int_{y \text{ constant}} \left(x + \frac{a^2y}{x^2 + y^2}\right)dx + \int y\,dy = c_1$$

or $\frac{x^2}{2} + a^2 y\left[\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right] + \frac{y^2}{2} = c_1$

or $x^2 + 2a^2 \tan^{-1}\left(\frac{x}{y}\right) + y^2 = c$, where $c = 2c_1$

## EXERCISE 7.6

In the following problems determine whether the given equation is exact. If exact, solve.

1. $x(2y^2 + 3x)dx + 2x^2y\,dy = 0$
2. $(ax+hy+g)dx + (hx+by+f)dy = 0$
3. $6xy^3\,dx + y^2(4y + 9x^2)dy = 0$
4. $(x + y)(x - y)dx + x(x - 2y)dy = 0$ - Not exact
5. $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$
6. $(ye^x - \sin x)dx - (e^x + 2y)dy = 0$
7. $x\,dy = (2xe^x - y + 6x^2)\,dx$
8. $\left(x^2y^3 - \frac{1}{1 + 9x^2}\right)dx + x^3y^2\,dy = 0$
9. $\left(3x^2 + \frac{2y}{x}\right)dx + \left(2\log 3x + \frac{3}{y}\right)dy = 0$
10. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]dy = 0$
11. $(3x^2 + 2y\sin 2x)dx + (2\sin^2 x + 3y^2)dy = 0$
12. $2x^2y\cos x\,dy - (x^2y^2\sin x - 2xy^2\cos x)dx = 0$
13. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$
14. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$
15. $(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0$

Solve each of the following equations subject to the indicated conditions.

16. $(xy^2 - 1)dx - (1 - x^2y)dy = 0$ ; $y = 1$, $x = 0$
17. $x\cos y\,dy = (2x - \sin y)dx$ ; $y(2) = 0$
18. $(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0$ ; $y(-2) = 1$
19. $(\cos x\sin x - xy^2)dx + y(1 - x^2)dy = 0$ ; $y(0) = 2$.
20. $y(\sin 2x + y\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0$ ; $y(0) = 3$

21. $\left(\frac{3-y}{x^2}\right)dx - \left(\frac{2x-y^2}{xy^2}\right)dy = 0 \quad ; \quad y(-1) = 2$

22. $(\sin 2x - \tan y)dx = x \sec^2 y\, dy \quad ; \quad y(\pi) = \frac{\pi}{4}$

23. $x(2\cos y + 3xy)dx - (y + x^2 \sin y - x^3)dy = 0 \quad ; \quad y(0) = 2.$

## 7.7 DIFFERENTIAL EQUATIONS REDUCIBLE TO EXACT FORM

### 7.7.1 INTEGRATING FACTORS

A differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

which is not exact can be transformed into an exact equation on multiplication by a suitable factor $\mu(x,y)$. Such a multiplier is known as **integrating factor** (I.F.) of the differential equation. For example, the equation $xdy - ydx = 0$ is not exact. But if we multiply by $\frac{1}{x^2}$, it becomes an exact differential equation.

$$\frac{xdy - ydx}{x^2} = 0 \quad \text{or} \quad d\left(\frac{y}{x}\right) = 0$$

whose solution is $\frac{y}{x} = c.$

Similarly, if we divide by $y^2$, we get $d\left(\frac{x}{y}\right) = 0$

leading to $\frac{x}{y} = c$ as the solution.

Again if we divide by xy, we obtain

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

or $$d[\log y] - d[\log x] = 0$$

or $$d\left[\log \frac{y}{x}\right] = 0$$

or $$\log \frac{y}{x} = c \Rightarrow \frac{y}{x} = e^c = k \quad \text{i.e.,} \quad y = kx$$

Unfortunately there is no single method for obtaining integrating factors. It can also be noticed that the number of integrating factors is infinite. In general, an integrating factor exists but in some cases it is likely to be difficult to determine it.

### 7.7.2 INTEGRATING FACTOR FOUND BY INSPECTION

In some cases, the integrating factor can be found by inspection only, after properly regrouping the terms of the differential equation and then recognising each group as being an exact differential of known function. In this connection, some experience and ingenuity combined with the following integrable combinations (exact differentials) of common occurrence will be found helpful in selecting suitable integrating factors and thereby in reducing a non-exact equation to an exact form. It is not a generally applicable method.

$$x\,dy + y\,dx = d(xy)$$

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$$

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{\dfrac{x\,dy - y\,dx}{x^2}}{\dfrac{x^2 + y^2}{x^2}} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{x\,dy + y\,dx}{xy} = d[\log(xy)]$$

$$\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{x\,dx \pm y\,dy}{x^2 \pm y^2} = \frac{1}{2}\,d[\log(x^2 \pm y^2)]$$

$$\frac{x\,dy - y\,dx}{x^2 - y^2} = \frac{1}{2}\,d\left[\log\frac{x+y}{x-y}\right]$$

$$\frac{x\,dy + y\,dx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$$

$$\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$$

$$\frac{2xy\,dy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$$

**EXAMPLE** Solve $\tan y\,dx + \tan x\,dy = 0$.

**SOLUTION** This equation is not exact. But on multiplying by $\cos x \cos y$ it becomes

$$\sin y \cos x\,dx + \sin x \cos y\,dy = 0$$

which is exact being expressible as $d[\sin y \sin x] = 0$

Hence the solution is $\sin y \sin x = c$.

## EXERCISE 7.7

Solve the following equations by the method of inspection.

1. $y\,dx - (x - 2y^3)dy = 0$
2. $x\,dy - y\,dx - \cos\frac{1}{x}dx = 0$
3. $(y^2e^x + 2xy)dx - x^2\,dy = 0$
4. $x\,dy - y\,dx = x^2y^3\,dx$
5. $x\,dy - y\,dx = (x^2 + y^2)dx$
6. $y(2xy + e^x)dx - e^x\,dy = 0$
7. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$
8. $x\,dx + y\,dy = 3y^2\sqrt{x^2 + y^2}\,dy$
9. $(2x^2y + e^x)y\,dx = (e^x + y^3)dy$
10. $y\,dx - x\,dy + (1 + x^2)dx + x^2 \sin y\,dy = 0$
11. $(x^2 + y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$
12. $(y + \log x)\,dx = xdy$
13. $y(x - 2y)\,dx = x(x - 3y)\,dy$
14. $x\,dx + \left(y - \sqrt{x^2 + y^2}\right)dy = 0$

## 7.8 RULES FOR THE DETERMINATION OF INTEGRATING FACTORS

It is not always possible to find the integrating factor of a differential equation by inspection. In that case we make use of some rules stated below (without proof) for finding integrating factors.

**RULE I** When the non-exact equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an integrating factor.

**EXAMPLE** Solve

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

**SOLUTION** This equation can be written as

$$(x^2y - 2xy^2)\,dx + (3x^2y - x^3)dy = 0 \qquad \ldots\ldots \text{(i)}$$

and is not exact since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Since the equation is homogeneous, and

$$Mx + Ny = (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$= x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0,$$

so by Rule I, $\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$

$\therefore$ Multiplying equation (i) by $\frac{1}{x^2y^2}$, we obtain

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$$

Now $\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$ and, therefore, the equation is exact and its solution is

$$\frac{x}{y} - 2\log x + 3\log y = c$$

or $$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

**Rule II** If the non-exact equation $M\,dx + N\,dy = 0$ is of the form

$y\,f(xy)\,dx + x\,g\,(xy)\,dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor.

**EXAMPLE** Solve

$$y\,(xy \sin xy + \cos xy)\,dx + x\,(xy \sin xy - \cos xy)\,dy = 0$$

**SOLUTION** Since $M = y\,f(xy)$ and $N = x\,g(xy)$ and

$$\begin{aligned} Mx - Ny &= y\,(xy \sin xy + \cos xy)\,x - x(xy \sin xy - \cos xy)\,y \\ &= x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin xy + xy \cos xy \\ &= 2xy \cos xy \\ &\neq 0 \end{aligned}$$

Hence $$I.F. = \frac{1}{Mx - Ny} = \frac{1}{2xy \cos xy}$$

Multiplying the given equation by this I.F., we get

$$\frac{1}{2}\left(\tan xy + \frac{1}{xy}\right)y\,dx + \frac{1}{2}\left(\tan xy - \frac{1}{xy}\right)x\,dy = 0$$

or $$\left(y \tan xy + \frac{1}{x}\right)dx + \left(x \tan xy - \frac{1}{y}\right)dy = 0$$

which is an exact equation since

$$\frac{\partial M}{\partial y} = \tan xy + xy \sec^2 xy = \frac{\partial N}{\partial x}$$

Hence its solution is

$$\int \left(y \tan xy + \frac{1}{x}\right)dx + \int \left(-\frac{1}{y}\right)dy = \log c$$

y constant

or $$\log \sec xy + \log x - \log y = \log c$$

$$\log \frac{x \sec xy}{y} = \log c$$

$$x \sec xy = cy$$

$$x = cy \cos xy$$

**Rule III** For the non–exact equation $Mdx + Ndy = 0$, if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say f(x), then I.F. $= e^{\int f(x)\,dx}$

**Rule IV** For the non–exact equation $Mdx + Ndy = 0$, if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say g(y), then $e^{\int g(y)dy}$ is an integrating factor.

**EXAMPLE** Solve

$$y\,dx - (y - 3x - 3)\,dy = 0$$

**SOLUTION** The given equation can be written as

$$y\,dx + (3 + 3x - y)\,dy = 0 \qquad \ldots (i)$$

Here $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 3$

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1 - 3}{3 + 3x - y} = \frac{-2}{3 + 3x - y}$ is not a function of x alone. But

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{3 - 1}{y} = \frac{2}{y}$ is a function of y alone.

Hence, I.F. $= e^{\int \frac{2}{y} dy} = e^{2\log y} = y^2$ $\{\because e^{\log m} = m\}$

Multiplying equation (i) by $y^2$, we get

$$y^3\,dx + (3y^2 + 3xy^2 - y^3)\,dy = 0$$

which is an exact equation as $\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x}$

Its solution is

$$xy^3 + \frac{3y^3}{3} - \frac{y^4}{4} = C$$

$$\text{or } xy^3 + y^3 - \frac{1}{4}y^4 = C$$

$$\text{or } y^3(4x - y + 4) = A, \text{ where } A = 4C$$

## EXERCISE 7.8

Using rules I – IV to find an I.F. solve

1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$
2. $y\,(xy+2x^2y^2)dx+x(xy-x^2y^2)dy = 0$
3. $(x^2 + y^2 + x)\,dx + xydy = 0$
4. $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)dy = 0$
5. $y\,(x + y + 1)\,dx + x\,(x + 3y + 2)\,dy = 0$
6. $(e^x - \sin y)\,dx + \cos y\,dy = 0$
7. $(y - xy^2)\,dx + xdy = 0$
8. $(y^2 + xy)\,dx = x^2\,dy$
9. $(3x^2y+2xy+y^3)dx + (x^2+y^2)dy = 0$
10. $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$
11. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$
12. $\dfrac{dy}{dx} = \dfrac{y\,(x + y)}{1 - 2y - x}$
13. $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$
14. $(2xy^4\,e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$
15. $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$
16. $(4xy+3y^2-x)dx + x(x+2y)dy = 0$
17. $(1 + xy)y\,dx + (1 - xy)xdy = 0$
18. $dx + \left(\dfrac{x}{y} - \sin y\right)dy = 0$
19. $(x^2 - 2x + 2y^2)dx + 2xy\,dy = 0$
20. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$

## 7.9 THE LINEAR EQUATION OF ORDER ONE

There is one important class of differential equations whose standard form has a frequent occurrence in practical applications. This is the class of linear equations of order one. We have already given the definition of the linear differential equation of order n. We now consider a first order ordinary differential equation linear in the dependent variable y and the independent variable x which is usually written in its standard form as

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \ldots\ldots (i)$$

On expressing (i) in the form $Mdx + Ndy = 0$ we have

$$[P(x)y - Q(x)]dx + dy = 0$$

Here $M = P(x)y - Q(x)$, $N = 1$

$\therefore \quad \frac{\partial M}{\partial y} = P(x)$, $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Thus equation (i) is not exact. We proceed to solve it by making its left hand side an exact differential, multiplying by a suitable integrating factor, say $\lambda(x)$ which is taken as a function of x alone and otain

$$\lambda(x)\frac{dy}{dx} + \lambda(x)P(x)y = \lambda(x)Q \quad \ldots\ldots \text{(ii)}$$

The term $\lambda(x)\frac{dy}{dx}$ suggests that the left hand side of (ii) can be obtained by differentiating the product $\lambda(x)\,y$. Hence we take

$$\lambda(x)\frac{dy}{dx} + \lambda(x)P(x)y = \frac{d}{dx}[\lambda(x)\,y]$$

$$= \lambda(x)\frac{dy}{dx} + y\frac{d}{dx}[\lambda(x)]$$

$$\Rightarrow \quad \frac{d}{dx}[\lambda(x)] = \lambda(x)P(x)$$

Separating the variables

$$\frac{1}{\lambda(x)}\,d\lambda = P(x)\,dx$$

Integrating

$$\log\lambda(x) = \int P(x)\,dx$$

or $\quad \lambda(x) = e^{\int P(x)\,dx}$

Hence $e^{\int P\,dx}$ is an integrating factor of (i).

On multiplying both sides of equation (i) by this integrating factor, we obtain

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)y\,e^{\int P(x)dx} = Q(x)\,e^{\int P(x)dx}$$

i.e., $\quad \frac{d}{dx}(ye^{\int P(x)dx}) = Qe^{\int P(x)dx}$

Integrating, we get

$$ye^{\int P(x)dx} = \int\{Qe^{\int P(x)dx}\}dx + c$$

which is the general solution of equation (i).

**NOTE** If the equation be $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y, then $\text{I.F.} = e^{\int P\,dy}$

and the solution will be

$$x \cdot e^{\int P\,dy} = \int [Qe^{\int P\,dy}]dy + c$$

**EXAMPLE 1** Solve

$$x\left(\frac{dy}{dx} + y\right) + y = 1$$

**SOLUTION** The given equation can be written as

$$\frac{dy}{dx} + y + \frac{y}{x} = \frac{1}{x}$$

or $$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}$$

where $P = 1 + \frac{1}{x}$ and $Q = \frac{1}{x}$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \left(1 + \frac{1}{x}\right) dx} = e^{x + \log x} = e^x e^{\log x} = xe^x$$

Hence the required solution is given by

$$y \times (\text{I.F.}) = \int [Q \times (\text{I.F.})]dx + c$$

i.e., $$y\,xe^x = \int \left(\frac{1}{x}\right) xe^x\,dx + c$$

$$= \int e^x\,dx + c$$

$$= e^x + c$$

or $$(yx - 1)e^x = c$$

**EXAMPLE 2** Solve

$$(1 + y^2)dx = (\tan^{-1}y - x)dy$$

**SOLUTION** We can write the equation as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \qquad \ldots\ldots \text{(i)}$$

which is linear with x as dependent variable.

$$\therefore \int P\,dy = \int \frac{1}{1+y^2}\,dy = \tan^{-1}y$$

$$\therefore \text{I.F.} = e^{\int P\,dy} = e^{\tan^{-1}y}$$

Multiplying both sides of equation (i) by I.F. the L.H.S. becomes the differential coefficient of x times $e^{\tan^{-1}y}$ with respect to y

$$\therefore \quad \frac{d}{dy}\left(xe^{\tan^{-1}y}\right) = e^{\tan^{-1}y} \times \frac{\tan^{-1}y}{1+y^2}$$

Integrating, we obtain

$$x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}\,dy + c$$

Put $\tan^{-1}y = t$ so that $\frac{1}{1+y^2}\,dy = dt$

Then the above equation gives

$$x\,e^{\tan^{-1}y} = \int e^t\,t\,dt + c$$

$$= te^t - \int 1 \cdot e^t\,dt + c \quad \text{[integrating by parts]}$$

$$= (t-1)e^t + c$$

$$= (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + c$$

$$\therefore \quad x = (\tan^{-1}y - 1) + c\,e^{\tan^{-1}y}$$

**EXAMPLE 3** Solve the initial value problem

$e^x[y - 3(e^x+1)^2]dx + (e^x+1)dy = 0, \quad y = 4$ when $x = 0$.

**SOLUTION** The given equation can be written as

$$\frac{dy}{dx} + \frac{e^x}{e^x+1}\,y = 3e^x(e^x+1)$$

Here $P = \frac{e^x}{e^x + 1}$ and $Q = 3e^x(e^x + 1)$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log(e^x + 1)} = e^x + 1$

Hence the solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

or $y(e^x + 1) = \int 3e^x(e^x + 1)^2 dx + c$

$$= (e^x + 1)^3 + c$$

But $y = 4$ when $x = 0$, therefore $c = 0$

Hence the required particular solution is

$$y = (e^x + 1)^2.$$

### 7.9.1 EQUATIONS REDUCIBLE TO LINEAR FORM

The differential equation

$$\frac{dy}{dx} + Py = Qy^n, \qquad \ldots\ldots \text{(i)}$$

where P and Q are functions of x alone and n is any real number except 0 and 1, is known as **Bernoulli's equation** after the Swiss mathematician Jacob Bernoulli (1654 — 1705). This equation is non-linear, but it can be transformed to linear form by suitable substitution and then the solution is obtained by the usual method of solving a linear equation.

Note that for $n = 0$, equation (i) reduced to the first order linear equation in standard form. When $n = 1$, it becomes $\frac{dy}{dx} + [P(x) - Q(x)]y = 0$ which is a separable equation and hence can be solved.

In order to solve (i), we divide both of its sides by $y^n$ and get

$$y^n \frac{dy}{dx} + Py^{1-n} = Q \qquad \ldots\ldots \text{(ii)}$$

Put $y^{1-n} = v$, $n \neq 0$, $n \neq 1$. Then on differentiating with respect to x,

$$(1 - n)y^n \frac{dy}{dx} = \frac{dv}{dx} \quad \Rightarrow \quad y^n \frac{dy}{dx} = \frac{1}{1 - n} \frac{dv}{dx}$$

Substituting in (ii) we obtain

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

or $$\frac{dv}{dx} + (1-n)Pv = (1-n)Q$$

which is a linear equation with v as dependent variable and can be solved for v by the method already discussed. Finally, replacement of v by $y^{1-n}$ will lead to a solution of (i).

**NOTE** (i) General equation reducible to linear form is $f'(y)\frac{dy}{dx} + Pf(y) = Q$, where P and Q are functions of x alone.

Putting $f(y) = v$ so that $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$, the given equation becomes

$$\frac{dv}{dx} + Pv = Q$$

which is linear in v and x and its solution is given by

$$v(I.F.) = \int Q(I.F.)dx + c$$

Replacing v by f(y) we obtain the required solution in terms of y and x.

(ii) The equation $\frac{dx}{dy} + Px = Qx^n$,

where P and Q are functions of y alone, is also Bernoulli's differential equation.

**EXAMPLE 1** Solve $x\frac{dy}{dx} + y = y^2 \log x$

**SOLUTION** Dividing the given equation by $xy^2$ we obtain

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{1}{x}\log x$$

Putting $\frac{1}{y} = v$ ; $-\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dv}{dx}$ it becomes

$$-\frac{dv}{dx} + \frac{1}{x}\cdot v = \frac{1}{x}\log x$$

$$\text{or } \frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{1}{x} \log x$$

which is linear in v.

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{-\log x} = \frac{1}{x}$$

Hence the solution is given by

$$v \cdot \frac{1}{x} = \int \frac{-1}{x^2} \log x \, dx + c$$

$$= \frac{1}{x} \log x - \int \frac{1}{x} \cdot \frac{1}{x} \, dx + c \qquad \text{(Integrating by parts)}$$

$$= \frac{1}{x} \log x + \frac{1}{x} + c$$

$$\text{or } v = \log x + 1 + cx = \log x + \log e + cx$$

$$\text{or } \frac{1}{y} = \log ex + cx$$

$$\text{or } y(cx + \log ex) = 1.$$

**EXAMPLE 2** Solve

$$\sec^2 y \frac{dy}{dx} + x \tan y = x^3 \qquad \ldots\ldots \text{(i)}$$

**SOLUTION** Put $v = \tan y$ so that $\frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$

Then (i) reduces to

$$\frac{dv}{dx} + x\, v = x^3$$

This equation is linear with v as dependent variable.

$$\therefore \text{ I.F.} = e^{\int x dx} = e^{\frac{x^2}{2}}$$

Hence the solution is

$$v\, e^{\frac{x^2}{2}} = \int x^3 \cdot e^{\frac{x^2}{2}} \, dx + c$$

Let $\frac{x^2}{2} = t$, $x\,dx = dt$. Then

$$v\,e^t = \int 2te^t\,dt = 2e^t(t-1) + c$$

or $$v = 2(t-1) + ce^{-t}$$

or $$\tan y = x^2 - 2 + ce^{-\frac{x^2}{2}}$$

**EXAMPLE 3** Solve the initial value problem

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}, \quad y(1) = 2$$

**SOLUTION** Multiplying the given equation by $y^3$, we get

$$y^3\frac{dy}{dx} + \frac{1}{2x}\cdot y^4 = x$$

Put $y^4 = v$ so that $4y^3\frac{dy}{dx} = \frac{dv}{dx}$. Then

$$\frac{1}{4}\cdot\frac{dv}{dx} + \frac{1}{2x}\cdot v = x$$

or $\frac{dv}{dx} + \frac{2}{x}\cdot v = 4x$, which is linear in v.

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{2\log x} = x^2$$

Hence the general solution is

$$v\,x^2 = \int (4x)\,x^2 dx + c = x^4 + c$$

or $$y^4x^2 = x^4 + c.$$

Applying $y(1) = 2$, we get $c = 15$.

Hence the required solution is

$$y^4x^2 = x^4 + 15 \quad \text{or} \quad y^4 = x^2 + 15x^{-2}.$$

## EXERCISE 7.9

Solve

1. $x\frac{dy}{dx} + 2y = \sin x$

2. $x\frac{dy}{dx} + (2x + 1)y = xe^{-2x}$

3. $x \log x \frac{dy}{dx} + y = 2 \log x$

4. $(x + 2y^3)\frac{dy}{dx} = y$

5. $\cos^3 x \frac{dy}{dx} + y \cos x = \sin x$

6. $\frac{dy}{dx}(x^2y^3 + xy) = 1$

7. $dr + (2r \cot \theta + \sin 2\theta)\, d\theta = 0$

8. $(x + 1)\frac{dy}{dx} - y = e^x(x + 1)^2$

9. $x^2(x^2 - 1)\frac{dy}{dx} + x(x^2 + 1)y = x^2 - 1$

10. $(1 + x^2)\frac{dy}{dx} + y = \arctan x$

11. $(1 - x^2)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$

12. $\frac{dy}{dx} + y = xe^{2x} y^3$

13. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

14. $(x - 1)^3\frac{dy}{dx} + 4(x - 1)^2 y = x + 1$

15. $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)e^x \sec y$

16. $x\frac{dy}{dx} - 3y = x - 1 \; ; \; y(1) = 0$

Solve the initial value problems:

17. $\frac{dy}{dx} + y = \sin x \; ; \; y(\pi) = 1$

18. $(x^2 + 1)\frac{dy}{dx} + 2xy = 4x^2 \; ; \; y(0) = 0$

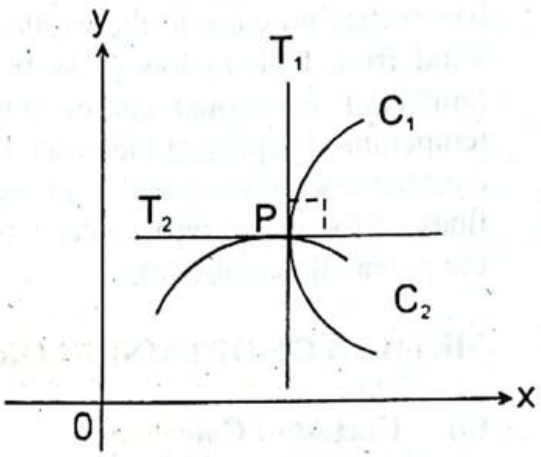
19. $\frac{dy}{dx} + 2y \tan x = \sin x \; ; \; y\left(\frac{\pi}{3}\right) = 0$

20. $\frac{dr}{d\theta} + r \tan \theta = \cos^2 \theta \; ; \; r\left(\frac{\pi}{4}\right) = 1$

21. $2xydy - (x^2 + y^2 + 1)dx = 0 \; ; \; y(1) = 1.$

## 7.9.2. ORTHOGONAL TRAJECTORIES

1. **Orthogonal Curves:** Two curves $C_1$ and $C_2$ are said to be **orthogonal** at a point P if and only if their tangent lines $T_1$ and $T_2$ at P intersect each other at right angles, i.e., the slopes of the tangents are negative reciprocals of each other.




2. A *trajectory* of a family of curves is a curve which cuts all the members of the family according to a given law.

Let $f(x, y, c) = 0,$ ..... (1)

where c is a paramater, be a family of curves.

If all the curves of (1) intersect orthogonally all the curves of another family of curves, say

$g(x, y, d) = 0$ ..... (2)

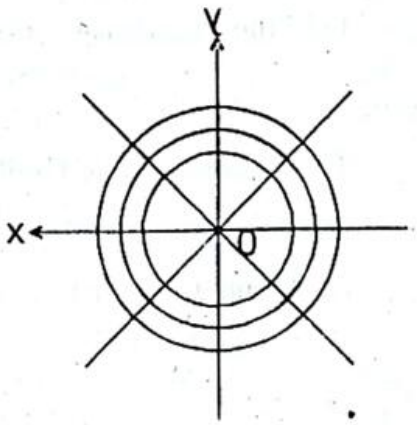
then the families (1) and (2) are said to be **orthogonal trajectories** of each other. In other words, an orthogonal trajectory is any one curve that intersects every curve of another family at right angles. To illustrate this we consider the following examples:

*(i)* The curve $y = -\frac{1}{3}x + 2$ is an orthogonal trajectory of the family $y = 3x + c$.

*(ii)* The members of the family of straight lines $y = mx$ through the origin and the family $x^2 + y^2 = k$ of concentric circles with centre at the origin constitute mutually orthogonal trajectories.

*(iii)* The family of circles $x^2 + y^2 + 2cy = 0$ through the origin with centres on the y-axis, constitute mutually orthogonal trajectories of the family of circles $x^2 + y^2 + 2kx = 0$ through the origin with centres on the x-axis.

**Note:** A family of curves that intersects a given family of curves at a specific constant angle $\alpha \neq \frac{\pi}{2}$ is called an *isogonal* family. The two families are said to be isogonal trajectories of each other.




Applications of orthogonal trajectories in physics and engineering are numerous. In fluid dynamics the stream lines and lines of constant velocity potential are orthogonal trajectories of each other. In a weather map the orthogonal trajectories of the family of isobars (i.e., the curves connecting all cities which report the same

barometric pressure to the weather bureau) would indicate the general direction of the wind from high to low pressure areas. Similarly, the orthogonal trajectories of the family of isothermal curves (i.e., the curves connecting points having the same temperature) represent the general direction of heat flow. Likewise, during the study of electricity we notice that in an electric field around two bodies of opposite charge the lines of force are perpendicular to the equipotential curves (i.e., curves along which the potential is constant).

## METHOD OF OBTAINING ORTHOGONAL TRAJECTORIES

### (a) Cartesian Coordinates

Suppose we want to find the orthogonal trajectories of the family of curves

$$f(x, y, c) = 0 \qquad \ldots\ldots (1)$$

First we form the differential equation of (1) by differentiating it and eliminating the parameter 'c' between the derived equation so obtained and (1).

Let the resulting differential equaiton be

$$g\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \ldots\ldots (2)$$

Then on replacing by its negative reciprocal $-\frac{dx}{dy}$ (which is slope of the orthogonal trajectories) in (2), we obtain

$$g\left(x, y, -\frac{dx}{dy}\right) = 0 \qquad \ldots\ldots (3)$$

the differential equaiton of orthogonal trajectories. The general solution of (3) gives the required family of orthogonal trajectories.

Let us illustrate the above procedure by some examples.

**Example 1**

Find the orthogonal trajectories of the family of rectangular hyperbolas $xy = c$.

**Solution**

The equation of the family of hyperbolas is

$$xy = c \qquad \ldots\ldots (1)$$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0, \qquad \ldots\ldots (2)$$

The general form of the differential equation of the first order and nth degree is

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \cdots + P_n = 0 \qquad \ldots\ldots (1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Three of the methods for solving such an equation are classified as under:

(i) equations solvable for p,

(ii) equations solvable for y, and

(iii) equations solvable for x.

## 7.10.1 EQUATIONS SOLVABLE FOR p

Since equation (1) is a polynomial of degree n in p we may, if possible, factorise it into n linear factors and write

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \cdots [p - f_n(x, y)] = 0$$

Equating each of these factors to zero, we get

$$p = f_1(x, y), \quad p = f_2(x, y), \cdots, \quad p = f_n(x, y)$$

Each of these equations is of the first order and first degree and can, therefore, be integrated by the methods already done. Let their solutions be

$$\phi_1(x, y, c_1) = 0, \quad \phi_2(x, y, c_2) = 0, \quad \cdots, \quad \phi_n(x, y, c_n) = 0,$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

These n solutions constitute the general solution of equation (1). The general solution may be combined as

$$\phi_1(x, y, c)\,\phi_2(x, y, c) \cdots \phi_n(x, y, c) = 0$$

**NOTE** There is no loss of generality if each of n arbitrary constants $c_1, c_2, \cdots, c_n$ is replaced by a single constant c as equation (1) is of first order and therefore, its general solution should contain only one arbitrary constant.

**EXAMPLE 1** Solve $x^2p^2 + xyp - 6y^2 = 0$, where $p = \dfrac{dy}{dx}$.

**SOLUTION** The given equation can be written as

$$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$$

or $$xp(xp + 3y) - 2y(xp + 3y) = 0$$

or $(xp + 3y)(xp - 2y) = 0$

either $xp + 3y = 0$

Replacing p by $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -3\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{3}{x}dx$$

Integrating, we get

$\log y = -3 \log x + \log c$

or $\log y = \log cx^{-3}$

$$\therefore \quad y = \frac{c}{x^3}$$

or $xp - 2y = 0$

Replacing p by $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\frac{dy}{y} = \frac{2}{x}dx$$

Integrating, we have

$\log y = 2 \log x + \log c$

$= \log cx^2$

$y = cx^2$

Hence the solution of given equation is

$$\left(y - \frac{c}{x^3}\right)(y - cx^2) = 0 \quad \text{or} \quad (yx^3 - c)(y - cx^2) = 0$$

**EXAMPLE 2** Solve $xp^2 + (y - 1 - x^2)p - x(y - 1) = 0$

**SOLUTION** We can re-arrange the given equation as

$xp^2 - x^2p + (y - 1)p - x(y - 1) = 0$

or $xp(p - x) + (y - 1)(p - x) = 0$

or $(p - x)(xp + y - 1) = 0$

Either $p - x = 0$ which gives $\frac{dy}{dx} = x \Rightarrow 2y - x^2 = c$ ..... (1)

or $xp + y - 1 = 0$

$$x\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = 1/x$$

which is linear in y.

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\log x} = x$$

$$\therefore \quad yx = \int \left(\frac{1}{x}\right) \cdot x \, dx + c$$

$$= x + c \qquad \ldots\ldots (2)$$

Combining (1) and (2), the solution of given equation is

$$(2y - x^2 - c)(xy - x - c) = 0$$

**EXAMPLE 3** Solve $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 = x^2$

**SOLUTION** We can write the equation as

$$x^2p^2 - 2xy\,p + 2y^2 - x^2 = 0$$

This is a quadratic in p and therefore,

$$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

which is homogeneous. Making the substitution $y = vx$, we have

$$v + x\frac{dv}{dx} = v \pm \sqrt{1 - v^2}$$

$$\text{or} \quad x\frac{dv}{dx} = \pm\sqrt{1 - v^2}$$

Separating the variables,

$$\frac{dv}{\sqrt{1 - v^2}} = \pm\frac{dx}{x}$$

Integrating both sides, we get

$$\sin^{-1} v = \pm \log x + c$$

or $\sin^{-1}\left(\frac{y}{x}\right) = \pm \log x + c$

Hence the required solution is

$$\left(\sin^{-1}\frac{y}{x} - \log x - c\right)\left(\sin^{-1}\frac{y}{x} + \log x - c\right) = 0$$

## EXERCISE 7.10

Solve

1. $p^2 - 7p + 12 = 0$
2. $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$
3. $p^2 - 2p \sinh x - 1 = 0$
4. $4xp^2 - (3x - a)^2 = 0$
5. $x^2p^2 + xp - y^2 - y = 0$
6. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$
7. $p^2 + 2py \cot x = y^2$
8. $yp^2 + (x - y^2)p - xy = 0$
9. $xy\,p^2 + p(3x^2 - 2y^2) - 6xy = 0$
10. $p(p - y) = x(x + y)$
11. $xy(p^2 + 1) = p(x^2 + y^2)$
12. $p^3 - p(x^2+xy+y^2) + xy(x+y) = 0$

## 7.11 EQUATIONS SOLVABLE FOR y

If the differential equation $\phi(x, y, p) = 0$ is solvable for y, it can be expressed as

$$y = f(x, p) \quad \ldots\ldots (1)$$

Differentiating (1) w.r.t. x and denoting $\frac{dy}{dx}$ by p we get an equation of the form

$$\frac{dy}{dx} = p = g\left(x, p, \frac{dp}{dx}\right) \quad \ldots\ldots (2)$$

This is an equation of the first order in two variables x and p. Let its solution be

$$\psi(x, p, c) = 0 \quad \ldots\ldots (3)$$

Elimination of p between equations (1) and (3) yields the required solution of the given equation involving x, y and c.

If elimination of p is not easily possible, then we may solve (1) and (3) for x and y in terms of p and obtain

$$x = \phi_1(p, c), \quad y = \phi_2(p, c) \qquad \ldots\ldots (4)$$

which constitute the parametric form of the required solution with p as a parameter.

**NOTE** This method is also useful for equations which do not contain x.

**EXAMPLE 1** Solve $p^2 + x^3p - 2x^2y = 0$

**SOLUTION** The equation is clearly solvable for y. We have

$$2y = xp + \frac{p^2}{x^2} \qquad \ldots\ldots (1)$$

Differentiating w.r.t. x and replacing $\frac{dy}{dx}$ by p, we get

$$2p = x\frac{dp}{dx} + p + p^2\left(-\frac{2}{x^3}\right) + \frac{2p}{x^2}\cdot\frac{dp}{dx}$$

or $$p\left(1 + \frac{2p}{x^3}\right) - x\frac{dp}{dx}\left(1 + \frac{2p}{x^3}\right) = 0$$

or $$\left(p - x\frac{dp}{dx}\right)\left(1 + \frac{2p}{x^3}\right) = 0 \qquad \ldots\ldots (2)$$

Considering $$p - x\frac{dp}{dx} = 0$$

we get $$\frac{dp}{p} = \frac{dx}{x}$$

Integrating $$\log p = \log x + \log c = \log cx$$

or $$p = cx \qquad \ldots\ldots (3)$$

To eliminate p we put $p = cx$ from (3) in (1) and get

$$2y = cx^2 + c^2$$

which is the required solution.

**NOTE** The factor $\left(1 + \frac{2p}{x^3}\right)$ of equation (2) has not been considered here as it does not contain the derivative $\frac{dp}{dx}$. If we eliminate p between $1 + \frac{2p}{x^3} = 0$ and (1) the resulting solution will not contain any arbitrary constant and hence cannot be the general solution. Its significance will be noted at a later stage under singular solution.

**EXAMPLE 2** Solve $y - 3xp = p^2$

**SOLUTION** We can write the given equation as

$$y = 3xp + p^2 \quad \ldots\ldots (1)$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 3x\frac{dp}{dx} + 3p + 2p\frac{dp}{dx}$$

or $$p = 3p + (3x + 2p)\frac{dp}{dx}$$

or $$-2p\frac{dx}{dp} = 3x + 2p$$

or $$\frac{dx}{dp} + \frac{3}{2p}x = -1$$

which is linear in x.

$$\therefore \quad \text{I.F.} = e^{\int \frac{3}{2p} dp} = e^{\frac{3}{2}\log p} = p^{3/2}$$

$$\therefore \quad xp^{3/2} = \int (-1) \cdot p^{3/2}\, dp + c = -\frac{2}{5}p^{5/2} + c$$

or $$x = -\frac{2}{5}p + cp^{-3/2} \quad \ldots\ldots (2)$$

Since it is not easy to eliminate p from (1) and (2), therefore, these two equations constitute the required solution with p as parameter.

## EXERCISE 7.11

Solve

1. $y + \log p = x$
2. $y = (x - a)p - p^2$
3. $xp^2 - 2py + ax = 0$
4. $y = 2xp + p^n$
5. $y = (1 + p)x + p^2$
6. $y = 2px + \tan^{-1}(xp^2)$
7. $y + px = p^2x^4$
8. $y = 2px + p^4x^2$
9. $y - 2px = f(xp^2)$
10. $p^3 + p = e^y$
11. $e^{n\,y} = p^2 - 1$
12. $y = p^2x + p$

## 7.12 EQUATIONS SOLVABLE FOR x

If the differential equation $\phi(x, y, p) = 0$ is solvable for x, we can express it as

$$x = f(y, p) \qquad \ldots\ldots (1)$$

Differentiating (1) w.r.t. to y, and denoting $\frac{dy}{dx}$ by p, we have

$$\frac{dx}{dy} = \frac{1}{p} = \psi\left(y, p, \frac{dp}{dy}\right) \qquad \ldots\ldots (2)$$

This is an equation of first order involving x and p. Let its solution be

$$F(y, p, c) = 0 \qquad \ldots\ldots (3)$$

Then on eliminating p from (1) and (3) we obtain the required solution of the given equation. In case it is not easy to eliminate p between (1) and (3), we express x and y in terms of p and c and get

$$x = \phi_1(p, c) \quad , \quad y = \phi_2(p, c) \qquad \ldots\ldots (4)$$

The two equations in (4) together constitute the required solution with p as a parameter.

NOTE This method is also useful for equations which do not involve y.

**EXAMPLE 1** Solve $p^3 - 4xy\,p + 8y^2 = 0$

**SOLUTION** The equation being solvable for x, we write

$$x = \frac{2y}{p} + \frac{p^2}{4y}$$

Differentiating w.r.t. y and replacing $\frac{dx}{dy}$ by $\frac{1}{p}$, we have

$$\frac{1}{p} = \frac{2}{p} + 2y\left(-\frac{1}{p^2}\right)\frac{dp}{dy} + \frac{p^2}{4}\left(-\frac{1}{y^2}\right) + \frac{2p}{4y}\frac{dp}{dy}$$

or $$\frac{2y}{p^2}\cdot\frac{dp}{dy} - \frac{1}{p} - \frac{p}{2y}\cdot\frac{dp}{dy} + \frac{p^2}{4y^2} = 0$$

or $$\left(\frac{2y}{p^2}\cdot\frac{dp}{dy} - \frac{1}{p}\right) - \frac{p^3}{4y^2}\left(\frac{2y}{p^2}\cdot\frac{dp}{dy} - \frac{1}{p}\right) = 0$$

or $$\left(\frac{2y}{p^2}\cdot\frac{dp}{dy} - \frac{1}{p}\right)\left(1 - \frac{p^3}{4y^2}\right) = 0$$

Ignoring $\left(1 - \frac{p^3}{4y^2}\right) = 0$, we obtain

$$\frac{2y}{p^2}\cdot\frac{dp}{dy} - \frac{1}{p} = 0 \Rightarrow \frac{2}{p}\,dp = \frac{1}{y}\,dy$$

Integration gives

$$2\log p = \log y + \log c = \log cy$$

or $$p^2 = cy \Rightarrow p = \sqrt{cy}$$

Substituting this value of p in the given equation, we have the p-eliminant as

$$\left(\sqrt{cy}\right)^3 - 4xy\left(\sqrt{cy}\right) + 8y^2 = 0$$

or $$\left(\sqrt{cy}\right)(cy - 4xy) = -8y^2$$

Squaring both sides, we obtain

$$cy\,(cy - 4xy)^2 = 64y^4$$

$$\Rightarrow \quad c\,(c - 4x)^2 = 64y$$

as the required solution

**EXAMPLE 2** Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$

**SOLUTION** We note the absence of y here. On solving for x the equation takes the form

$$x = \tan^{-1} p + \frac{p}{1+p^2} \qquad \ldots\ldots (1)$$

Differentiating w.r.t. y, we have

$$\frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{(1+p^2) - p\cdot 2p}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

$$= \left[\frac{1}{1+p^2} + \frac{1-p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$= \frac{1+p^2+1-p^2}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

or $dy = \frac{2p}{(1+p^2)^2}\cdot dp = (1+p^2)^{-2}\cdot 2p\, dp$

Integrating, we get $y = c - \frac{1}{1+p^2}$ ..... (2)

Equation (1) together with (2) constitutes the required solution with p as parameter.

## EXERCISE 7.12

Solve

1. $x - y = p^2$
2. $y = 2px + 4p^2y$
3. $x = y + a \log p$
4. $xp^3 = a + bp$
5. $p^2 - xp + 1 = 0$
6. $(2x - m)p = y(1 - np^2)$
7. $y = 2px + y^2p^3$
8. $p^3 + x = p(y + 3)$
9. $4yp = y^4 - 4xp^2$
10. $y^2 \log y - xy\, p = p^2$

## 7.13 CLAIRAUT'S EQUATION

The differential equation of the form

$$y = px + f(p) \qquad \ldots\ldots \text{(i)}$$

where p stands for $\frac{dy}{dx}$ is known as Clairaut's equation, named after the mathematician A.C. Clairaut (1713 – 65) who first solved this equation.

Equation (i) can be treated either as solvable for y or solvable for x. Differentiating it w.r.t. x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Either $x + f'(p) = 0$, {which will lead to singular solution and hence is discarded}

$$\text{or} \quad \frac{dp}{dx} = 0 \Rightarrow p = c \qquad \ldots\ldots \text{(ii)}$$

Eliminating p between (i) and (ii), we have

$$y = cx + f(c)$$

which gives the general solution of Clairaut's equation. Thus we notice that the solution to Clairaut's equation is obtained simply by replacing p by an arbitrary constant c in it.

**EXAMPLE 1** Solve $3p = \cos(y - px)$

**SOLUTION** We can re-write the given equation as

$$\cos^{-1}(3p) = y - px$$

$$\text{or} \quad y = px + \cos^{-1}(3p)$$

which is Clairaut's equation.

Hence the required general solution is

$$y = cx + \cos^{-1}(3c)$$

**EXAMPLE 2** Solve $(x - a)p^2 + (x - y)p - y = 0$

**SOLUTION** Re-arranging the given equation, we have

$$xp^2 - ap^2 + xp - yp - y = 0$$

or $$y(p + 1) = xp^2 + xp - ap^2$$

$$= xp(p + 1) - ap^2$$

$$\therefore \quad y = xp - \frac{a p^2}{p + 1}$$

which is Clairaut's equation.

Hence its general solution is

$$y = cx - \frac{a c^2}{c + 1}$$

## 7.13.1 EQUATIONS REDUCIBLE TO CLAIRAUT'S FORM

There are certain differential equations which can be converted to Clairaut's form by means of suitable substitutions. For example,

(i) The equations of the type

$$y^2 = pxy + f\left(\frac{py}{x}\right)$$

can be reduced to Clairaut's form by making use of the substitution

$$x^2 = u \quad \text{and} \quad y^2 = v$$

(ii) The equations of the type

$$e^{by}(a - bp) = f(pe^{by - ax})$$

can be transformed to Clairaut's form by the substitutions

$$e^{ax} = u, \quad e^{by} = v$$

**EXAMPLE 1** Reduce $(px - y)(py + x) = a^2p$ to Clairaut's form by making use of the substitution $x^2 = u$, $y^2 = v$.

**SOLUTION** Since $x^2 = u$, $y^2 = v$ ..... (1)

Therefore, $2xdx = du$, $2ydy = dv$

Thus $$p = \frac{dy}{dx} = \frac{x}{y} \cdot \frac{dv}{du}$$

Substituting this value of p, the given equation reduces to

$$\left(\frac{x^2}{y} \cdot \frac{dv}{du} - y\right)\left(x\frac{dv}{du} + x\right) = a^2 \cdot \frac{x}{y} \cdot \frac{dv}{du}$$

or $$\left(x^2\frac{dv}{du} - y^2\right)\left(\frac{dv}{du} + 1\right) = a^2\frac{dv}{du}$$

Denoting $\frac{dv}{du}$ by P, we obtain

$$(x^2P - y^2)(P + 1) = a^2P$$

or $$(uP - v)(P + 1) = a^2P$$

or $$y = uP - \frac{a^2P}{P + 1}$$

This is Clairaut's form and hence by changing P to c the general solution is

$$v = cu - \frac{a^2c}{c + 1}$$

or $$y^2 = cx^2 - \frac{a^2c}{c + 1}$$ ..... by (1)

**EXAMPLE 2** Solve $e^{3x}(p - 1) + p^3 e^{2y} = 0$

**SOLUTION** Re-arranging the equation, we get

$$e^{3x}(p - 1) = -p^3e^{2y}$$

$$1 - p = p^3 e^{2y-3x}$$

Multiplying both sides by $e^y$, we have

$$e^y(1 - p) = p^3 e^{3y-3x} = (p e^{y-x})^3$$

Let $e^x = u$, $e^y = v$

so that $e^x dx = du$, $e^y dy = dv$

Hence $p = \frac{dy}{dx} = \frac{e^x}{e^y} \cdot \frac{dv}{du} = \frac{u}{v} \cdot \frac{dv}{du} = \frac{u}{v} \cdot P$ where $P = \frac{dv}{du}$

Substituting for p, the given equation becomes

$$u^3\left(\frac{u}{v} \cdot P - 1\right) + \frac{u^3}{v^3} \cdot P^3 \cdot v^2 = 0$$

$$uP - v + P^3 = 0$$

or $$v = uP + P^3$$

which is Clairaut's form. Hence the required general solution is

$$v = uc + c^3$$

or $$e^y = ce^x + c^3$$

## EXERCISE 7.13

Solve

1. $y + p^2 = p(x - a)$
2. $p = \log(px - y)$
3. $y = x\frac{dy}{dx} + a\frac{dx}{dy}$
4. $py = p^2(x - m) + n$
5. $(y - px)(p - 1) = p$
6. $\sin px \cos y = \cos px \sin y + p$
7. $p(xp - y + k) - 3 = 0$
8. $p^2(x^2 - 4) - 2pxy + y^2 - 4 = 0$

Using suitable substitutions, reduce the following differential equations to Clairaut's form and hence solve:

9. $x^2(y - px) = yp^2$
10. $y - 2xp = f(xp^2)$
11. $(x - py)\left(x - \frac{y}{p}\right) = a^2$
12. $e^{4x}(p - 1) + e^{2y}p^2 = 0$
13. $y = yp^2 + 2px$
14. $2yp = xp^2 + ax$
15. $p^2\cos^2 y + p \sin x \cos x \cos y - \sin y \cos^2 x = 0$

## 7.14 SINGULAR SOLUTIONS

So far we have been finding the general solution and the particular solution obtainable by assigning particular values to the arbitrary constants involved in the general solution. But other than the general and particular solutions, a differential equation may possess a solution which does not involve any arbitrary constant and, in general, is not obtainable from the general solution by assigning particular values to the arbitrary constants. Such a solution is called **singular solution**. For example, consider the differential equation

$$\left(\frac{dy}{dx}\right)^2 - y = 0 \qquad \ldots\ldots \text{(i)}$$

$$\Rightarrow \quad \left(p + \sqrt{y}\right)\left(p - \sqrt{y}\right) = 0, \text{ where } p = \frac{dy}{dx}$$

Either $p + \sqrt{y} = 0 \Rightarrow 2\sqrt{y} + x + c = 0$

or $p - \sqrt{y} = 0 \Rightarrow 2\sqrt{y} - x - c = 0$

Thus the general solution of (i) is

$$\left[2\sqrt{y} + (x + c)\right]\left[2\sqrt{y} - (x + c)\right] = 0$$

or $$4y = (x + c)^2 \qquad \ldots\ldots \text{(ii)}$$

which is a family of parabolas. We observe that in addition to (ii), (i) is also satisfied by

$$y = 0 \qquad \ldots\ldots \text{(iii)}$$

which is not included in the general solution (ii) and is, therefore, a singular solution. By eliminating c between

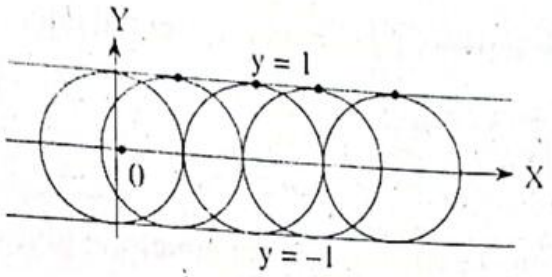
$$f(x, y, c) \equiv 4y - (x + c)^2 = 0$$

and $$\frac{\partial f}{\partial c} \equiv -2(x + c) = 0$$

we get $y = 0$ which shows that the singular solution given by (iii) is, in fact, the envelope of the parabolas (ii).

**NOTE** A curve which at each of its points is tangent to some one of the curves of a one-parameter family is called an **envelope** of that family. For example (see figure), the envelope of the circles

$$(x - c)^2 + y^2 = 1$$

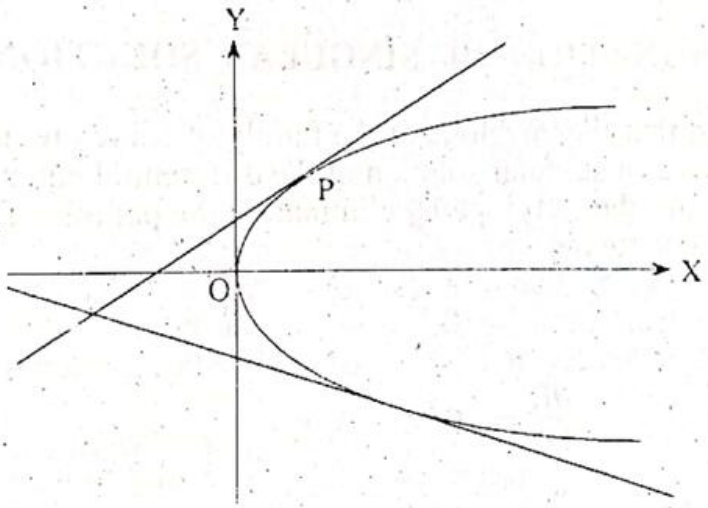



From differential calculus we know that the general method for finding the envelope is to eliminate the parameter c from the family of curves

$$f(x, y, c) = 0 \quad \text{and} \quad \frac{\partial f}{\partial c} = 0.$$

Again from coordinate geometry we know that the straight line

$$y = px + \frac{a}{p} \quad \text{where } m = \frac{dy}{dx} = p$$




touches the parabola $y^2 = 4ax$ for all values of the arbitrary parameter p. At P(x, y) the point of contact of any particular tangent and the parabola, the tangent and parabola have the same values of x, y and $p = \frac{dy}{dx}$. Hence the equation of the parabola $y^2 = 4ax$ must be a solution of the differential equation $y = px + \frac{a}{p}$.

But we can verify that $y^2 = 4ax$ is the envelope of the family of lines $y = px + \frac{a}{p}$. Hence the envelope represents a singular solution of the differential equation as it does not contain any new arbitrary constant and is not deducible from the general solution. In general, if the family of curves

$$f(x, y, c) = 0 \qquad \ldots\ldots \text{(iv)}$$

presenting the complete primitive of a certain differential equation

presenting the complete primitive of a certain differential equation

$$\phi\left(x,\ y,\ \frac{dy}{dx}\right) = 0, \qquad \ldots\ldots \text{(v)}$$

possesses an envelope, then the equation of the envelope gives the singular solution of the differential equation (v).

Let the family of curves possess an envelope and P(x, y) be any point on this envelope. There exists a curve of the family, say $f(x, y, c_1) = 0$ which touches the envelope at P(x, y). The values of x, y, $\frac{dy}{dx}$ for the curve at P satisfy the given differential equation. Also the values of x, y, $\frac{dy}{dx}$ at P for the envelope are the same as for the curve. Thus we see that the values of x, y, $\frac{dy}{dx}$ at every point of the envelope satisfy the given differential equation. Hence the equation of the envelope is a singular solution of the differential equation.

## 7.14.1 DETERMINATION OF SINGULAR SOLUTIONS

We have shown that the envelope of the family of curves represented by the complete primitive gives a singular solution of the differential equation. Hence we proceed for determining the envelope by eliminating the parameter c between the equation of the family of curves

$$f(x, y, c) = 0 \qquad \ldots\ldots \text{(vi)}$$

and

$$\frac{\partial f}{\partial c} = 0 \qquad \ldots\ldots \text{(vii)}$$

Let this eliminant be

$$g(x, y) = 0 \qquad \ldots\ldots \text{(viii)}$$

As the eliminant given by (viii) may represent loci other than the envelope, we must verify whether any part of the locus represented by (viii) is or is not a solution of the differential equation. Singular solution does satisfy the given differential equation.

**NOTE** (i) In case $f(x, y, c) = 0$ is of the form $Ac^2 + Bc + C = 0$, then the envelope is given by $B^2 - 4AC = 0$ ..... (ix)

(ii) The eliminant given by (viii) or by (ix) is known as **c-discriminant.**

## 7.14.2 THE p-DISCRIMINANT METHOD

We shall now consider the method of obtaining the singular solution of a differential equation directly from the equation itself, without finding the complete primitive.

Now (viii) represents the locus of points (x, y) such that at least two of the corresponding values of c are equal i.e., such that at least two of the curves of the family through (x, y) coincide. Since the equation

$$\phi(x, y, p) = 0 \qquad \ldots\ldots \text{(x)}$$

determines the slopes of the tangents to the curves of the family through (x, y), we see that for a point (x, y) satisfying (viii) at least two of the corresponding values of p must be equal. Thus the envelope and hence the singular solution is also contained in the locus obtained by eliminating p between (x) and

$$\frac{\partial \phi}{\partial p} = 0$$

**NOTE**

(i) When p is eliminated from the parametric equations $\phi(x,y,p) = 0$ and $\frac{\partial \phi}{\partial p} = 0$, the resulting equation is called the **p-discriminant** equation and does not contain an arbitrary constant.

(ii) The p-discriminant for the quadratic $Lp^2 + Mp + N = 0$, where L, M, N are any functions of x, y, is given by $M^2 - 4LN = 0$.

(iii) When an equation leading to singular solution is obtained during the natural course of solving an equation, the p-discriminant equation furnishes us with a **check** and confirmation.

(iv) It can also be shown that the p-discriminant relation may sometimes represent other loci besides the envelope. The part of this relation that satisfies the differential equation will be the singular solution.

**REMARK** Consider the Clairaut's equation

$$y = px + f(p) \qquad \ldots\ldots (1)$$

Its general solution is

$$y = cx + f(c) \qquad \ldots\ldots (2)$$

where c is an arbitrary constant.

Differentiating (2) partially w.r.t. c, we get

$$0 = x + f'(c) \qquad \ldots\ldots (3)$$

Then the singular solution is obtained by eliminating c from equations (2) and (3).

Now differentiating (1) partially w.r. t. p, we have

$$0 = x + f'(p) \quad \ldots\ldots (4)$$

If we eliminate the parameter p between (1) and (4) the resulting eliminant will be the same as the c-eliminant obtained from (2) and (3). Thus we see that in Clairaut's form of the differential equation both p and c-discriminants are the same.

**EXAMPLE 1** Solve and find the singular solution, if it exists.

$$p^2 + 2px^3 - 4x^2y = 0 \quad \ldots\ldots (1)$$

**SOLUTION** Solving the given equation for y, we get

$$4y = 2px + \frac{p^2}{x^2}$$

Differentiating w.r.t. x, we have

$$4p = 2p + 2x\frac{dp}{dx} + \frac{1}{x^2}\cdot 2p\frac{dp}{dx} - \frac{2}{x^3}p^2$$

$$\text{or} \quad \frac{dp}{dx}\left(2x + \frac{2p}{x^2}\right) = 2p + \frac{2p^2}{x^3}$$

$$\text{or} \quad \frac{dp}{dx}\left(2x + \frac{2p}{x^2}\right) = \frac{p}{x}\left(2x + \frac{2p}{x^2}\right) \quad \ldots\ldots (2)$$

$$\text{or} \quad \frac{dp}{dx} = \frac{p}{x} \Rightarrow \frac{dp}{p} = \frac{dx}{x} \Rightarrow p = cx \quad \ldots\ldots (3)$$

Eliminating p between (1) and (3) we get the general solution as

$$c^2x^2 + 2cx^4 - 4x^2y = 0$$

$$\text{or} \quad c^2 + 2cx^2 - 4y = 0$$

which is a quadratic in c. Hence

c-discriminant is $4x^2 + 16y = 0$ or $x^4 + 4y = 0$

Also p-disc. is $4x^6 + 16x^2y = 0$ or $4x^2(x^4 + 4y) = 0$

Since $x^4 + 4y = 0$ appears once in both c and p discriminants and satisfies (1), hence it is the required singular solution.

**NOTE**

(i) $x = 0$ is an extraneous locus.

(ii) The elimination of p between the discarded factor $2x + \frac{2p}{x^2} = 0$ of (2) and the given equation (1) also leads to the equation $x^4 + 4y = 0$ of singular solution. It furnishes an **alternate method** for finding singular solution. The presence of such a factor implies the existence of a S.S. but not conversely. Hence some authors do not recommend the use of this procedure in finding singular solutions.

**EXAMPLE 2** Find the general solution and the singular solution, if any:

$$xp^2 + (x - y)\,p + (1 - y) = 0 \qquad \ldots\ldots (1)$$

**SOLUTION** We can re-arrange the given equation as

$$xp^2 + xp - yp - y + 1 = 0$$

or $$xp\,(p + 1) - y\,(p + 1) + 1 = 0$$

or $$xp - y + \frac{1}{p + 1} = 0$$

or $$y = px + \frac{1}{p + 1}$$

which is Clairaut's form of the differential equation. Hence the general solution is

$$y = cx + \frac{1}{c + 1}$$

or $$xc^2 + (x - y)\,c + (1 - y) = 0 \qquad \ldots\ldots (2)$$

The c-discriminant of (2) is

$$(x - y)^2 - 4x\,(1 - y) = 0$$

which is coincident with p-discriminant of (1) and hence gives the required S.S. and can be re-written as

$$(x - y)^2 + 4xy = 4x \Rightarrow (x + y)^2 = 4x.$$

**EXAMPLE 3** Find the general solution of

$$y + yp^2 = 2xp \qquad \ldots\ldots (1)$$

and examine for singular solutions.

**SOLUTION** Solving the given equation for x, we have

$$2x = \frac{y}{p} + yp$$

Differentiating w.r.t. y, we get

$$\frac{2}{p} = \frac{1}{p} + y\left(-\frac{1}{p^2}\frac{dp}{dy}\right) + y\frac{dp}{dy} + p$$

or $$\frac{1}{p} - p = \left(y - \frac{y}{p^2}\right)\frac{dp}{dy}$$

or $$\frac{1-p^2}{p} = y\left(\frac{p^2-1}{p^2}\right)\frac{dp}{dy}$$

or $$p + y\frac{dp}{dy} = 0$$

or $$\frac{dp}{p} + \frac{dy}{y} = 0 \Rightarrow py = c \quad \ldots\ldots (2)$$

Eliminating p between (2) and the given equation (1), we have the general solution as

$$y + y\left(\frac{c}{y}\right)^2 = 2x\left(\frac{c}{y}\right)$$

or $$c^2 - 2xc + y^2 = 0$$

Its c-discriminant is $x^2 - y^2 = 0$

Also p-discriminant is $x^2 - y^2 = 0$

As both $y = x$ and $y = -x$ satisfy the give equation, they give singular solutions (a pair of straight lines).

## EXERCISE 7.14

Solve and examine for singular solution.

1. $4p^2(x - a) = 1$
2. $4xp^2 = (3x - 1)^2$
3. $p^3 + px - y = 0$
4. $(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0$
5. $xp^2 - 2yp + x = 0$
6. $y = 2px + y^2p^3$

7. $3xy = 2px^2 - 2p^2$

8. $(x^2 - 4)\,p^2 - 2xyp - x^2 = 0$

9. $6p^2y^2 + 3px - y = 0$

10. $p^2 + x^3p - 2x^2y = 0$

11. $x^2(y - px) = p^2y$

12. Show that the equation

$$(px^2 + y^2)(px + y) = (p + 1)^2$$

can be reduced to Clairaut's form by means of the substitutions $u = xy$, $v = x + y$. Hence solve the equation and find its singular solution.

13. Solve the equation

$$x^2p^2 + y(2x + y)p + y^2 = 0,$$

using the substitutions $y = u$, $xy = v$ and find its singular solution.

Investigate for singular solutions by finding the p-discriminant:

14. $x^4p^2 - xp - y = 0$

15. $xp^2 - py - y = 0$

16. $e^p - p + x(y - 1) - 1 = 0$

17. $p^4 - 2p^2 + 1 - y^4 = 0$

18. $p^3 - 2x^2p + 4xy = 0$

# 8

# DIFFERENTIAL EQUATIONS – II

## 8.1 LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Linear differential equations constitute the most important single class of ordinary differential equations because of their wide range of application in physical problems. In the previous chapter we discussed methods of solving linear ordinary differential equations of the first order. This chapter deals with the linear differential equations of higher orders. Most of the methods of solving these equations are due to Euler and D'Alembert.

We have already discussed that in a linear differential equation the dependent variable and all its derivatives appear only in the first degree and are not multiplied together. The general linear nth-order differential equation is of the form

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = X, (p_0 \neq 0) \quad \ldots (1)$$

where X and the coefficients $p_0, p_1, p_2, \cdots, p_n$ are constants or functions of x.

If X is not identically zero, then the linear equation (1) is said to be non-homogeneous; whereas the equation

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = 0$$

obtained by setting $X = 0$ in (1) is said to be its associated or related homogeneous equation.

For example, the equation

$$4x^3 y''' + 3xy'' - 2y' + 6y = \sin x$$

is a non-homogeneous linear third-order ordinary differential equation and

$$y'' - 3y' + 8y = 0$$

is a homogeneous linear second-order differential equation.

**NOTE** Here the word **homogeneous** does not have the same sense as in case of homogeneous linear equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

of first order, where the coefficients are homogeneous functions.

### 8.1.1 LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

We shall now consider in detail the linear differential equations with constant coefficients. If $p_0, p_1, p_2, \cdots, p_n$ are all constants (X may or may not be a constant), then equation (1) is called a Linear Differential equation with constant coefficients. If $p_0 \neq 0$, we can make the leading coefficient unity by dividing each term of the equation by this coefficient and obtain the equation in the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots\ldots \quad (2)$$

where $a$'s are all constants.

It may be observed that the coefficient of highest order derivative $\frac{d^n y}{dx^n}$ which has been made unity, does not introduce any loss of generality.

### 8.1.2 LINEAR OPERATOR

An **operator** A is defined if there is a rule which assigns to each function u of a certain class another function $v = Au$. For example, the symbol $\frac{d}{dx}$ of differentiation with respect to x is an operator because on applying it on a function we get a new function after differentiation.

An operator A is called a **linear operator** if for any functions $u_1$, $u_2$ and any constants $c_1$, $c_2$ we have

$$A(c_1 u_1 + c_2 u_2) = c_1 Au_1 + c_2 Au_2.$$

$\frac{d}{dx}$ is an example of a linear operator.

In general, if $u_1, u_2, \ldots, u_n$ are n functions and $c_1, c_2 \cdots, c_n$ are n constants, then

$$A(c_1 u_1 + c_2 u_2 + \cdots + c_n u_n) = c_1 Au_1 + c_2 Au_2 + \ldots + c_n Au_n.$$

The function $u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ is called a linear combination of the functions $u_1, u_2, \cdots, u_n$.

### 8.1.3 DEFINITION

The sum of two operators A and B is an operator A + B defined by

$$(A + B)\, u = Au + Bu$$

If A and B are linear operators, then

$$\begin{aligned}(A + B)(c_1u_1 + c_2u_2) &= A(c_1u_1 + c_2u_2) + B(c_1u_1 + c_2u_2)\\ &= c_1 Au_1 + c_2 Au_2 + c_1 Bu_1 + c_2 Bu_2 \ (\because A, B \text{ are linear})\\ &= c_1(Au_1 + Bu_1) + c_2(Au_2 + Bu_2)\\ &= c_1(A + B)u_1 + c_2(A + B)u_2\end{aligned}$$

which establishes that the sum of linear operators is a linear operator.

For the sake of convenience we usually write

$$\frac{d}{dx} \equiv D \text{ so that } Dy = \frac{dy}{dx} = y'(x)$$

Similarly, we write

$$\frac{d^2}{dx^2} \equiv D^2, \ldots, \frac{d^n}{dx^n} \equiv D^n.$$

The symbol D is called a **differential operator**. Using this notation the differential equation (2) can be expressed in terms of the symbol D as

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)\, y = X$$

$$\text{i.e., } f(D)\, y = X \qquad \ldots (3)$$

where the polynomial $f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n$ is an nth-order differential operator.

We can show that the operator D can be treated as an algebraic quantity and all the algebraic operations such as factorisation, resolution into partial fractions, expansion etc. can be performed on the polynomial f(D), provided that the coefficients of all powers of D are constants. Following are some important properties of the polynomial differential operators;

**(i)** $A + B = B + A$ **(ii)** $(A + B) + C = A + (B + C)$

(iii) $AB = BA$ (iv) $(AB)C = A(BC)$

(v) $A(B + C) = AB + AC$ (vi) $A = B \Leftrightarrow Ay = By$

### 8.1.4 THEOREM

If f(D) be a polynomial operator, then

$f(D)\, e^{mx} = e^{mx} f(m)$ , where m is a constant.

**PROOF** Let $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$ ..... (4)

Then $f(D)\, e^{mx} = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)\, e^{mx}$

$= D^n e^{mx} + a_1 D^{n-1} e^{mx} + a_2 D^{n-2} e^{mx} + \cdots + a_{n-1} D e^{mx} + a_n e^{mx}$

But $D e^{mx} = m e^{mx}$

$D^2 e^{mx} = m^2 e^{mx}$

$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots$

$D^n e^{mx} = m^n e^{mx}$

where n is a positive integer.

Therefore,

$f(D)\, e^{mx} = m^n e^{mx} + a_1 m^{n-1} e^{mx} + a_2 m^{n-2} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx}$

$= e^{mx} [m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n]$

$= e^{mx} f(m)$ ..... from (4)

which proves the theorem.

**COROLLARY** If m is a root of the equation $f(m) = 0$, then

$f(D)\, e^{mx} = e^{mx} \cdot 0 = 0$

### 8.1.5 EXPONENTIAL SHIFT FORMULA

If f(D) is a polynomial operator in D with constant coefficients, then

$e^{ax} f(D)\, y = f(D - a)\, [e^{ax} y]$

or $e^{ax} f(D + a)\, y = f(D)\, [e^{ax} y]$

**PROOF** In order to prove this result we first consider the effect of the operator $(D-a)$ on the product $e^{ax}\,y$. Now

$$\begin{aligned}(D-a)\,[e^{ax}\,y] &= D(e^{ax}\,y) - a\,e^{ax}\,y \\ &= e^{ax}\,Dy + a\,e^{ax}\,y - a\,e^{ax}\,y \\ &= e^{ax}\,Dy.\end{aligned}$$

Similarly,

$$\begin{aligned}(D-a)^2\,[e^{ax}\,y] &= (D-a)\,(e^{ax}\,Dy) \\ &= D(e^{ax}\,Dy) - a\,e^{ax}\,Dy \\ &= e^{ax}\,D^2y + a\,e^{ax}\,Dy - a\,e^{ax}\,Dy \\ &= e^{ax}\,D^2y\end{aligned}$$

Continuation of this process of operating n-times yields

$$(D-a)^n\,[e^{ax}\,y] = e^{ax}\,D^n y$$

If f(D) is a polynomial in D with constant coefficients, then by using the linearity of differential operators we obtain

$$f(D-a)\,[e^{ax}\,y] = e^{ax}\,f(D)y$$

$$\text{i.e.,}\quad e^{ax}\,f(D)\,y = f(D-a)\,[e^{ax}\,y]$$

The above relation tells us the method of shifting an exponential factor from the left of a differential operator to the right of the operator.

## 8.1.6 SOLUTION OF HOMOGENEOUS LINEAR D.E.

If in equation (3), X is not identically zero so that the equation is inhomogeneous, then the corresponding homogeneous equation is

$$f(D)\,y = 0 \qquad \ldots (5)$$

A simple but useful statement is that an equation which is already known to be linear, is homogeneous if and only if the function $y = 0$ is a solution (trivial solution). This follows from the fact that if the operator f(D) is linear,

$$f(D)\,[0] = f(D)\,[0 \cdot y] = 0 \cdot f(D)\,y = 0.$$

An important property of linear homogeneous differential equation is exposed in the next theorem.

### 8.1.7 THEOREM

Any linear combination of two or more solutions is also its solution. That is,

If $y_1, y_2, \cdots, y_n$ are n linearly independent* solutions of the homogeneous linear nth-order differential equation $f(D)y = 0$ in some given domain and if $c_1, c_2, \cdots, c_n$ are arbitrary constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution of the equation in the same domain.

**PROOF** Since y's are solution of $f(D)\,y = 0$

so $f(D)y_1 = 0$, $f(D)y_2 = 0, \ldots, f(D)\,y_n = 0$

Hence $f(D)y = f(D)\,[c_1y_1 + c_2y_2 + \cdots + c_ny_n]$

$= c_1\,f(D)y_1 + c_2\,f(D)y_2 + \cdots + c_n\,f(D)y_n$

$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0$

$= 0$

Thus y is the general or complete solution of the equation $f(D)\,y = 0$, because it contains n arbitrary constants.

Note that the above theorem is true even if the coefficients are functions of x instead of constants.

**COROLLARY** If $y = y_1(x)$ is a solution of a homogeneous linear differential equation, then a constant multiple $y = c_1y_1(x)$ of this solution is also a solution.

**NOTE** The above theorem is known as **Superposition Principle**. The superposition principle and its special case given as corollary, are properties that non-linear differential equations, in general, do not possess. We can verify that $y = \frac{1}{x}$ is a solution of the non-linear differential equation $y'' = 2y^3$ on the interval $(0, \infty)$ but the constant multiple $y = \frac{c}{x}$ is not a solution of the equation when $c \neq 0, \pm 1$.

**EXAMPLE** It can be verified that the functions $u_1 = e^x$, $u_2 = e^{2x}$, and $u_3 = e^{3x}$ all satisfy the homogeneous equation

$$u''' - 6u'' + 11u' - 6u = 0.$$

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* i.e., we cannot find constants $a_1, a_2, \cdots, a_n$ such that $a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0$ identically (i.e., for all x)

Hence by the above theorem another solution is

$u = c_1e^x + c_2e^{2x} + c_3e^{3x}$, where c's are arbitrary constants.

## 8.1.8 SOLUTION OF NON-HOMOGENEOUS EQUATIONS

We now divert our attention to the solution of a non-homogeneous linear nth-order differential equation (3). Any function $y_p(x)$ not containing arbitrary constants and satisfying (3) is called a **particular solution** or a **particular integral** (P. I.) of this equation. Also the linear combination

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

which has already been proved to be the general solution of the associated homogeneous equation (5), is known as the **complementary function** (C.F.) for the non-homogeneous equation (3).

We shall now prove that the general solution of a non-homogeneous linear differential equation is

$y$ = complementary function + any particular integral

= C.F. + P.I.

## 8.1.9 THEOREM

If $y = y_c(x)$ is the complete solution of the homogeneous linear differential equation $f(D)y = 0$ and $y = y_p(x)$ is a particular solution of the nonhomogeneous equation $f(D)y = X$, then

$$y = y_c(x) + y_p(x)$$

is the general (complete) solution of the non-homogeneous equation $f(D)y = X$.

**PROOF** If $y = y_c$ is the complete solution of $f(D)y = 0$,

then $f(D)y_c = 0$ ..... (6)

Again $y = y_p$ is a particular solution of $f(D)y = X$,

Therefore, $f(D)y_p = X$ ..... (7)

Adding equations (6) and (7), we obtain

$$f(D)[y_c + y_p] = X$$

That is, $y = y_c + y_p$ satisfies the equation $f(D)y = X$.

Hence it is the complete solution because it contains n arbitrary constants in $y_c$.

**NOTE** The above theorem is true for variable coefficients as well.

## 8.1.10 AUXILIARY EQUATION

Consider the nth-order homogeneous linear differential equation with constant coefficients

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

Its symbolic form is

$$f(D)y = 0$$

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$

is linear differential operator.

By corollary of theorem 8.1.4, if m is the root of $f(m) = 0$, then $f(D)\, e^{mx} = 0$.

Comparing it with $f(D)y = 0$, we see that $y = e^{mx}$ is a solution of $f(D)y = 0$.

It is interesting to note that all solutions of this equation are exponential functions or constructed out of exponential functions.

The equation $f(m) = 0$ is called the **auxiliary equation** or **characteristic equation** associated with $f(D)y = 0$ and is obtained by putting $D = m$ in $f(D) = 0$

**NOTE** If we compare $f(m) = 0$ with $f(D) = 0$ we observe that the characteristic equation $f(m) = 0$ will give the same values of m as the equation $f(D) = 0$ gives of D. Therefore, we can treat $f(D) = 0$ as our characteristic equation and solve it for D considering D as an algebraic quantity like m.

The auxiliary equation

$$f(m) = m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n = 0$$

is an algebraic equation of degree n. Let the n roots determined by it be $m_1, m_2, m_3, \cdots, m_n$. Then

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}, \cdots, y_n = e^{m_n x}$$

are solutions of $f(D)y = 0$.

The roots of the auxiliary equation may be

(i) real and distinct

(ii) real and repeated

(iii) complex conjugate

(iv) complex repeated

Since the solution of the differential equation f(D) y = 0 depends upon the nature of roots of the auxiliary equation, so we will discuss the above cases one by one.

**CASE I** If the n roots $m_1, m_2, \cdots, m_n$ of the auxiliary equation f(m) = 0 are all real and different, then the solutions

$$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \cdots, y_n = e^{m_n x}$$

are linearly independent and, therefore, the general solution of the equation f(D)y = 0 is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \quad \ldots (8)$$

where the constants $c_1, c_2, \cdots, c_n$ are arbitrary.

**EXAMPLE 1** Solve the initial value problem

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0 \text{ where } y = 1, \frac{dy}{dx} = 2 \text{ at } x = 0.$$

**SOLUTION** The auxiliary equation associated with the given equation is

$$m^2 + 5m - 6 = 0$$

Its roots are $m_1 = 1$ and $m_2 = -6$ which are real and distinct.

Hence the general solution is

$$y = c_1 e^x + c_2 e^{-6x}$$

where the constants $c_1$ and $c_2$ are arbitrary.

Since $y = 1$ at $x = 0$, therefore, $c_1 + c_2 = 1$

and $\frac{dy}{dx} = 2$ at $x = 0$ gives $c_1 - 6c_2 = 2$

These two equations yield $c_1 = \frac{8}{7}$ and $c_2 = -\frac{1}{7}$

Hence the solution of the given problem is

$$y = \frac{8}{7} e^x - \frac{1}{7} e^{-6x}$$

**EXAMPLE 2** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

**SOLUTION** We can write the equation in symbolic form as

$$(D^2 - 4D + 1)\, y = 0$$

Then the corresponding characteristic equation is

$$m^2 - 4m + 1 = 0$$

$$\therefore \quad m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Hence the general solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$$

**REMARK** The above solution can also be written in the form

$$y = e^{2x}\left[c_1 e^{\sqrt{3}\,x} + c_2 e^{-\sqrt{3}\,x}\right]$$

$$= e^{2x}\left[c_1(\cosh\sqrt{3}\,x + \sinh\sqrt{3}\,x) + c_2(\cosh\sqrt{3}\,x - \sinh\sqrt{3}\,x)\right]$$

$$= e^{2x}\left[(c_1 + c_2)\cosh\sqrt{3}\,x + (c_1 - c_2)\sinh\sqrt{3}\,x\right]$$

$$= e^{2x}\left[A\cosh\sqrt{3}x + B\sinh\sqrt{3}x\right],$$

where $A = c_1 + c_2$ and $B = c_1 - c_2$

In general, if the roots of the characteristic equation are $\alpha \pm \sqrt{\beta}$, then the G.S. is given by

$$y = e^{\alpha x}\left[c_1 \cosh\sqrt{\beta}\,x + c_2 \sinh\sqrt{\beta}\,x\right]$$

**CASE II** The auxiliary equation $f(m) = 0$ of the differential equation $f(D)\,y = 0$ may have some repeated roots. Let two of the n roots, say $m_1$ and $m_2$ be equal. Then the solution given by (8) can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$= de^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}, \text{ where } d = c_1 + c_2.$$

This solution contains only $(n-1)$ arbitrary constants and therefore, it cannot be the general solution of a nth-order differential equation. To obtain complete solution in this case, the given equation $f(D)y = 0$ can be written as

$$(D - m_1)(D - m_1)(D - m_3) \cdots (D - m_n)\, y = 0$$

For the repeated factor, the corresponding part of the solution is the solution of the equation

$$(D - m_1)(D - m_1)\, y = 0$$

Putting $(D - m_1)\, y = v,$ ..... (9)

we get $(D - m_1)\, v = 0$

or $$\frac{dv}{dx} - m_1 v = 0$$

which is a first order linear equation in v with its

$$\text{I.F.} = e^{\int P dx} = e^{\int -m_1 dx} = e^{-m_1 x}$$

and hence its solution is

$$v \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x}\, dx + B = B$$

or $$v = B e^{m_1 x}$$

Substituting this value of v in (9), we obtain

$$(D - m_1)y = B e^{m_1 x}$$

or $$\frac{dy}{dx} - m_1 y = B e^{m_1 x}$$

which is linear in y. Hence its solution is

$$y e^{-m_1 x} = \int B e^{m_1 x} \cdot e^{-m_1 x}\, dx + A$$

$$= Bx + A$$

or $$y = (A + Bx) e^{m_1 x}$$

Thus the complete solution of the equation $f(D)y = 0$ is given by

$$y = (A + Bx) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Similarly if the characteristic equation has three equal roots, each equal to $m_1$, then the part of the solution corresponding to the repeated factor is given by

$$e^{m_1 x}(A + Bx + Cx^2)$$

and therefore, the complete solution of the given equation $f(D)y = 0$ is

$$y = (A + Bx + Cx^2)e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

and so on for the roots repeated any number of times.

**EXAMPLE 3** Solve $\frac{d^4y}{dx^4} - 8\frac{d^2y}{dx^2} + 16y = 0$

**SOLUTION** Symbolic form of the given equation is

$$(D^4 - 8D^2 + 16)y = 0$$

Therefore, its auxiliary equation is

$$D^4 - 8D^2 + 16 = 0$$

or $$(D^2 - 4)^2 = 0$$

or $$(D^2 - 4)(D^2 - 4) = 0$$

$\therefore$ $$D = \pm 2, \pm 2$$

Thus 2 and –2 are both double roots of the characteristic equation. Hence the complete solution of the given equation, by Case II, is given by

$y = (c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{-2x}$, the c's being arbitrary constants.

**EXAMPLE 4** Solve the initial-value problem

$y''' - 6y'' + 9y' = 0$; $y(0) = 0$, $y'(0) = 2$ and $y''(0) = -6$

**SOLUTION** The symbolic form of the given equation is

$$(D^3 - 6D^2 + 9D)y = 0$$

The auxiliary equation $m^3 - 6m^2 + 9m = 0$ gives

$$m(m^2 - 6m + 9) = 0$$

either $m = 0$ or $(m - 3)^2 = 0 \Rightarrow m = 3, 3$

Hence the general solution of the given equation is

$$y = c_1 e^{0x} + e^{3x}(c_2 + c_3 x) = c_1 + e^{3x}(c_2 + c_3 x)$$

Thus $y' = e^{3x}(3c_2 + c_3 + 3c_3 x)$

and $y'' = e^{3x}(9c_2 + 6c_3 + 9c_3 x)$

To find the particular solution satisfying the given initial conditions we make the following substitutions :

$y(0) = 0$ which yields $c_1 + c_2 = 0$ ..... (i)

$y'(0) = 2$ which leads to $3c_2 + c_3 = 2$ ..... (ii)

$y''(0) = -6$ which implies $9c_2 + 6c_3 = -6$ ..... (iii)

Solving (i), (ii) and (iii) simultaneously, we obtain

$c_1 = -2$ , $c_2 = 2$ and $c_3 = -4$

Substituting these values of the c's in the general solution we get the required particular solution as

$$y = -2 + e^{3x}(2 - 4x)$$

**CASE III** Since the characteristic equation $f(m) = 0$ is an algebraic equation with real coefficients, therefore, if its roots be complex they will occur in conjugate pairs. Suppose that the roots $m_1$ and $m_2$ are two such complex roots, then

$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ ($\alpha$, $\beta$ are real and $\beta \neq 0$)

Since the roots are different, by case I, the corresponding part of the solution of the equation $f(D)y = 0$, takes the form

$$c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x}[c_1 e^{i\beta x} + c_2 e^{-i\beta x}]$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2(\cos \beta x - i \sin \beta x)]$$

{By Euler's Theorem : $e^{i\theta} = \cos \theta + i \sin \theta$}

$$= e^{\alpha x}[(c_1 + c_2)\cos \beta x + i(c_1 - c_2)\sin \beta x]$$

$$= e^{\alpha x}[A_1 \cos \beta x + A_2 \sin \beta x]$$ , where $A_1 = c_1 + c_2$ and $A_2 = (c_1 - c_2)\, i$

Therefore, the C.S. of the given equation $f(D)\, y = 0$ is

$$y = e^{\alpha x}[A_1 \cos \beta x + A_2 \sin \beta x] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Note that if there are two pairs of complex roots $\alpha \pm i\beta$ and $\gamma \pm i\delta$ then the general solution is

$$y = e^{\alpha x}[A_1 \cos \beta x + A_2 \sin \beta x] + e^{\gamma x}[A_3 \cos \delta x + A_4 \sin \delta x]$$

$$+ c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

**CASE IV** Now we discuss the case when auxiliary equation $f(m) = 0$ has repeated complex roots. Suppose it has two pairs of complex and repeated roots $\alpha \pm i\beta$, $\alpha \pm i\beta$. That is, let $m_1 = \alpha + i\beta$ be repeated twice and $m_2 = \alpha - i\beta$ be also repeated twice. Then by case II, the corresponding part of the complete solution is

$$(A_1 x + A_2) e^{(\alpha + i\beta)x} + (A_3 x + A_4) e^{(\alpha - i\beta)x}$$

which as before in case III, by using Euler's theorem and simplification, is transformed to the form

$$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$$

Therefore, the complete solution of the given equation $f(D)y = 0$ is

$$y = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$$

Similarly, if the auxiliary equation has $\alpha + i\beta$ and $\alpha - i\beta$ as conjugate complex roots, each repeated r-times, then the corresponding part of the general solution of $f(D)\, y = 0$ will be

$$y = e^{\alpha x}[(c_1 + c_2 x + \cdots + c_r x^{r-1}) \cos \beta x + (c_{r+1} + c_{r+2} x + \cdots + c_{2r} x^{r-1}) \sin \beta x]$$

where $c_1, c_2, \cdots, c_r, c_{r+1}, \cdots, c_{2r}$ are arbitrary constants.

**EXAMPLE 5** Solve $\dfrac{d^2y}{dx^2} + y = 0$ subject to $y\left(\dfrac{\pi}{3}\right) = 2, y'\left(\dfrac{\pi}{3}\right) = -4.$

**SOLUTION** The auxiliary equation $D^2 + 1 = 0$ has the imaginary roots $0 \pm i$

$\therefore$ the general solution of the given equation is

$$y = e^{0x}[c_1 \cos x + c_2 \sin x] = c_1 \cos x + c_2 \sin x$$

Differentiating, we get

$$y' = \frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$$

Now $y\left(\frac{\pi}{3}\right) = 2 \Rightarrow c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3} = 2$

$$\Rightarrow c_1\left(\frac{1}{2}\right) + c_2\left(\frac{\sqrt{3}}{2}\right) = 2$$

$$\Rightarrow c_1 + \sqrt{3}\, c_2 = 4 \quad \ldots\ldots \text{(i)}$$

and $y'\left(\frac{\pi}{3}\right) = -4 \Rightarrow -c_1 \sin\frac{\pi}{3} + c_2 \cos\frac{\pi}{3} = -4$

$$\Rightarrow -c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = -4$$

$$\Rightarrow -\sqrt{3}\, c_1 + c_2 = -8 \quad \ldots\ldots \text{(ii)}$$

Solving (i) and (ii) simultaneously, we obtain

$c_1 = 1 + 2\sqrt{3}$ and $c_2 = -2 + \sqrt{3}$

Hence the solution of the given problem is

$$y = \left(1 + 2\sqrt{3}\right)\cos x + \left(-2 + \sqrt{3}\right)\sin x$$

**EXAMPLE 6** Solve $\frac{d^4y}{dx^4} + 4y = 0$

**SOLUTION** Symbolic form of the given equation is

$$(D^4 + 4)\, y = 0$$

Corresponding characteristic equation is

$$D^4 + 4 = 0$$

or $(D^4 + 4D^2 + 4) - 4D^2 = 0$

or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

Thus $D = \frac{-2 \pm \sqrt{4-8}}{2}$ or $D = \frac{2 \pm \sqrt{(-2)^2-8}}{2}$

i.e., $D = -1 \pm \sqrt{-1}\ ,\quad 1 \pm \sqrt{-1}$

$= -1 \pm i\ ,\quad 1 \pm i$

Hence the complete solution of the given equation is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$$

## EXERCISE 8.1

Find the general solution of each of the following differential equations (primes denote derivatives w.r.t.x)

1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

2. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$

3. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0$

4. $y''' + 3y'' - 4y = 0$

5. $\frac{d^5y}{dx^5} - 16\frac{dy}{dx} = 0$

6. $y''' - y'' + y' - y = 0$

7. $y''' - 4y'' + y' + 6y = 0$

8. $(D^6 + D^5 - D^4 - D^3)\,y = 0$

9. $y''' - 3y'' + 9y' + 13y = 0$

10. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

11. $(D^4 - 5D^2 + 4)\,y = 0$

12. $(D^4 + 8D^2 + 16)\,y = 0$

13. $(D^4 - 2D^2 - 3)\,y = 0$

14. $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$

Solve the initial-value problems:

15. $y'' + y' - 2y = 0$ ; $y(0) = 3$, $y'(0) = 0$

16. $y'' - 6y' + 9y = 0$ ; $y(0) = 2$, $y'(0) = 1$

17. $y'' - 10y' + 28y = 0$ ; $y(0) = 1$, $y'(0) = 0$

18. $y'' + 4y' + 5y = 0$ ; $y(0) = 1$, $y'(0) = 0$

19. $y''' - 6y'' + 11y' - 6y = 0$ ; $y(0) = 0$ ; $y'(0) = 0$, $y''(0) = 2$

20. $(D^3 + 7D^2 + 19D + 13)\,y = 0$ ; $y(0) = 0$, $y'(0) = 2$, $y''(0) = -12$

## 8.2 SOLUTION OF NON-HOMOGENEOUS LINEAR D.E.

We have already seen that the general solution $y = y_c + y_p$ of the non-homogeneous differential equation $f(D)\, y = X$, where $f(D)$ is a linear polynomial differential operator with constant coefficients, consists of the sum of complementary function and the particular integral. The complementary function of this equation is given by the complete solution of the associated equation $f(D)\, y = 0$ and the methods to obtain C.F. have also been discussed. There are may methods like the method of undetermined coefficients or the method of variation of parameters whereby particular solution can be obtained. But in this section we shall concentrate our attention on the study of the method of inverse operator which is relatively easy and short in comparison to other methods.

### 8.2.1 THE INVERSE OPERATOR $\frac{1}{f(D)}$

**DEFINITION**

$\frac{1}{f(D)}\, X$ is defined to be that function of x, not involving any arbitrary constant, which when operated upon by $f(D)$ yields X.

$$\text{i.e., } f(D)\left[\frac{1}{f(D)}\, X\right] = X$$

In other words $\frac{1}{f(D)}\, X$ satisfies the equation $f(D)y = X$ and is, therefore, its particular solution written symbolically as

$$y_p = \frac{1}{f(D)}\, X$$

From the above definition we observe that $f(D)$ and $\frac{1}{f(D)}$ (also denoted by $f^{-1}(D)$) are inverse operators.

Also in particular $\frac{1}{D}\, X = \int X\, dx$

Let $\frac{1}{D}\, X = y$ ..... (i)

Operating on both sides by D, we get

$$D\left[\frac{1}{D}\, X\right] = Dy \quad \Rightarrow X = \frac{dy}{dx}$$

$\therefore \quad y = \int X\,dx$ , no arbitrary constant being added since (i) does not contain any constant.

### 8.2.2 GENERAL METHOD FOR DETERMINING PARTICULAR INTEGRAL

We first prove a result in the form of a theorem which shall be useful in the determination of the particular integral.

### 8.2.3 THEOREM

If X is a function of x or a constant, then

$$\frac{1}{D-m}X = e^{mx}\int X e^{-mx}\,dx$$

**PROOF** Let $y = \frac{1}{D-m}X$

Operating on both sides by $D-m$, we obtain

$$(D-m)y = X$$

or

$$\frac{dy}{dx} - my = X$$

This is a linear first order differential equation. Its

$$\text{I.F.} = e^{\int -m\,dx} = e^{-mx}$$

Hence its solution is given by

$$ye^{-mx} = \int X e^{-mx}\,dx + \text{an arbitrary constant.}$$

But we are interested in finding the particular integral so the arbitrary constant is omitted. Therefore,

$$y = e^{mx}\int X e^{-mx}\,dx$$

or

$$\frac{1}{D-m}X = e^{mx}\int e^{-mx} X\,dx$$

**COROLLARY** $\frac{1}{D+m}X = e^{-mx}\int e^{mx} X\,dx$

**8.2.4** The above theorem enables us to evaluate the particular integral $\frac{1}{f(D)}X$ as follows:

**(i) METHOD OF FACTORS** If f(D) is a polynomial of degree n in D, it can be resolved into n factors. Let

$$f(D) \equiv (D - m_1)(D - m_2) \cdots (D - m_n)$$

Now the particular integral

$$\frac{1}{f(D)} X = \frac{1}{D - m_1} \cdot \frac{1}{D - m_2} \cdots \frac{1}{D - m_n} X$$

$$= \frac{1}{D - m_1} \cdot \frac{1}{D - m_2} \cdots \frac{1}{D - m_{n-1}} \cdot e^{m_n x} \int X e^{-m_n x} dx$$

Then we evaluate

$$\frac{1}{D - m_{n-1}} \left\{ e^{m_n x} \int X e^{-m_n x} dx \right\}$$

Continuing this process for the n symbolic factors in the denominator from right to left we obtain the required particular integral.

**(ii) ALTERNATE METHOD (by Partial Fractions)**

Suppose that the operator $\frac{1}{f(D)}$ is resolved into partial fractions. Then

$$\frac{1}{f(D)} X = \frac{1}{(D-m_1)(D-m_2) \cdots (D-m_n)} X$$

$$= \left( \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n} \right) X$$

$$= A_1 e^{m_1 x} \int e^{-m_1 x} X \, dx + A_2 e^{m_2 x} \int e^{-m_2 x} X \, dx + \cdots + A_n e^{m_n x} \int e^{-m_n x} X \, dx$$

**NOTE** The above general method can be used to evaluate particular integral in any problem. But it is recommended to be used when X is of the forms tan *ax*, cot *ax*, sec *ax*, cosec *ax*. Similarly the method of partial fractions should be given preference when the factors are all distinct.

**EXAMPLE 1** Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$

**SOLUTION** Symbolic form of this equation is

$$(D^2 - 3D + 2) y = e^{3x}$$

The characteristic equation

$$D^2 - 3D + 2 = 0$$

$$\Rightarrow \quad (D - 1)(D - 2) = 0$$

$$\Rightarrow \quad D = 1, 2.$$

Hence the C.F. is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

Now the particular integral $y_p$ is given by

$$y_p = \frac{1}{(D-1)(D-2)} e^{3x}$$

$$= \frac{1}{D-1} \cdot e^{2x} \int e^{-2x} \cdot e^{3x} \, dx$$

$$= \frac{1}{D-1} \cdot e^{2x} \cdot e^x$$

$$= \frac{1}{D-1} e^{3x}$$

$$= e^x \int e^{-x} e^{3x} \, dx$$

$$= e^x \cdot \frac{e^{2x}}{2}$$

$$= \frac{e^{3x}}{2}$$

If we apply the method of partial fractions, then

$$y_p = \frac{1}{(D-1)(D-2)} e^{3x} = \left(\frac{1}{D-2} - \frac{1}{D-1}\right) e^{3x}$$

$$= e^{2x} \int e^{-2x} e^{3x} \, dx - e^x \int e^{-x} e^{3x} \, dx = e^{2x} e^x - e^x \cdot \frac{e^{2x}}{2} = \frac{e^{3x}}{2}$$

Hence the complete solution in either case is $y = c_1 e^x + c_2 e^{2x} + \frac{e^{3x}}{2}$

**EXAMPLE 2** Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$

**SOLUTION** The characteristic equation is

$$D^2 + a^2 = 0 \Rightarrow D = \pm ia$$

$$\therefore \quad y_c = c_1 \cos ax + c_2 \sin ax$$

and
$$y_p = \frac{1}{D^2+a^2} \sec ax = \frac{1}{(D+ia)(D-ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

Now $\frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$

$$= e^{iax} \int (\cos ax - i \sin ax) \cdot \frac{1}{\cos ax} dx, \text{ ref. 1.6.}$$

$$= e^{iax} \int (1 - i \tan ax) \, dx$$

$$= e^{iax}\left[x + \frac{i}{a} \log \cos ax\right]$$

Replacing i by –i, we obtain

$$\frac{1}{D+ia} \sec ax = e^{-iax}\left[x - \frac{i}{a} \log \cos ax\right]$$

Hence the particular integral becomes

$$y_p = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a} \log \cos ax\right\} - e^{iax}\left\{x - \frac{i}{a} \log \cos ax\right\}\right]$$

$$= \frac{x}{a}\left(\frac{e^{iax} - e^{iax}}{2i}\right) + \frac{1}{a^2} \log \cos ax \left(\frac{e^{iax} + e^{iax}}{2}\right)$$

$$= \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax$$

Hence the general solution is

$$y = y_c + y_p$$

$$\text{or } y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax.$$

## EXERCISE 8.2

Solve the differential equations.

1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$

2. $\frac{d^2y}{dx^2} + a^2y = \tan ax$

3. $\frac{d^3y}{dx^3} - \frac{dy}{dx} = e^x$

4. $(D^2 + 2D + 1)y = 2e^{2x}$

## 8.3 SHORT METHODS FOR FINDING P.I.

The general method to obtain the particular integral becomes laborious due to successive integrations and lot of calculations. Depending upon the special form of the function X, we shall now discuss and develop the methods, shorter than the general methods, of obtaining the particular integral $\frac{1}{f(D)} X$ of the differential equation $f(D)y = X$ where

$$f(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots + P_{n-1} D + P_n$$

and all the P's are constants.

**8.3.1 CASE I. P.I.** When X is of the form $e^{ax}$ where '$a$' is any constant.

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{provided} \quad f(a) \neq 0$$

**PROOF** By simple differentiation, we have

$$D(e^{ax}) = ae^{ax}, D^2(e^{ax}) = a^2e^{ax}, \cdots, D^n(e^{ax}) = a^ne^{ax}.$$

Multiplying $P_n$ by $e^{ax}$, $P_{n-1}$ by $D(e^{ax})$, $P_{n-2}$ by $D^2(e^{ax})$, ….. , $P_1$ by $D^{n-1}(e^{ax})$, 1 by $D^n(e^{ax})$

and adding, we obtain

$$(D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n) e^{ax} = (a^n + P_1a^{n-1} + \cdots + P_{n-1} a + P_n) e^{ax}$$

$$\text{or} \quad f(D)e^{ax} = f(a) e^{ax}$$

$$\therefore \quad \frac{1}{f(D)} [f(D) e^{ax}] = \frac{1}{f(D)} [f(a) e^{ax}] = f(a) \frac{1}{f(D)} e^{ax}$$

or $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, provided $f(a) \neq 0$ (i.e., '$a$' is not a root of $f(D) = 0$)

NOTE If $X$ = a constant = $\lambda$, say, then

$$\frac{1}{f(D)} \lambda = \lambda \frac{1}{f(D)} e^{0x} = \frac{\lambda}{f(0)}, \text{ provided } f(0) \neq 0$$

### 8.3.2 CASE OF FAILURE

If '$a$' is a root of $f(D) = 0$ i.e., if $f(a) = 0$, then

P.I. $= \frac{1}{f(D)} e^{ax}$ = is infinite.

and the above method fails and we proceed as under.

Since $D = a$ is a root of $f(D) = 0$, therefore $(D - a)$ is at least one factor of $f(D)$. Let $D = a$ be a root of multiplicity r (root repeated r-times) so that $(D-a)^r$ is a factor of $f(D)$. Then $f(D) = (D-a)^r \phi(D)$, where $\phi(a) \neq 0$.

Therefore, $$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{(D-a)^r} \cdot \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{(D-a)^r} \cdot \frac{1}{\phi(a)} e^{ax}, \text{ for } \phi(a) \neq 0$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{(D-a)^r} e^{ax} \quad \ldots\ldots \text{(i)}$$

But $$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{-ax} \cdot e^{ax} \, dx = e^{ax} \int 1 \cdot dx = xe^{ax} \quad \ldots\ldots \text{(ii)}$$

$$\frac{1}{(D-a)^2} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{D-a} e^{ax}$$

$$= \frac{1}{D-a} \cdot xe^{ax} \quad \ldots\ldots \text{by (ii)}$$

$$= e^{ax} \int e^{-ax} xe^{ax} \, dx = e^{ax} \cdot \frac{x^2}{2}$$

$$= \frac{x^2}{\underline{2}} e^{ax} \quad \ldots\ldots \text{(iii)}$$

$$\frac{1}{(D-a)^3} e^{ax} = \frac{1}{(D-a)} \cdot \frac{1}{(D-a)^2} e^{ax}$$

$$= \frac{1}{D-a} \cdot \frac{x^2}{\underline{|2}} e^{ax} \quad \ldots\ldots \text{ by (iii)}$$

$$= e^{ax} \int e^{-ax} \cdot \frac{x^2}{\underline{|2}} e^{ax} \, dx$$

$$= \frac{x^3}{\underline{|3}} e^{ax}$$

Continuing in this manner, finally we have

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\underline{|r}} e^{ax}$$

Substituting in (i), the particular integral is given by

$$\frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{\underline{|r}} e^{ax}, \text{ where } f(D) = (D-a)^r \phi(D) \quad \ldots\ldots (iv)$$

### 8.3.3 ALTERNATE FORM OF THE ABOVE FORMULA

We have $f(D) = (D-a)^r \phi(D)$

Differentiating r-times w.r.t.D, we obtain

$$f^r(D) = \underline{|r} \phi(D) + \text{terms containing } (D-a) \text{ and its higher powers}$$

$$\therefore \quad f^r(a) = \underline{|r} \phi(a) \Rightarrow \phi(a) = \frac{f^r(a)}{\underline{|r}}$$

Hence by substituting in (iv) we get

$$\frac{1}{f(D)} e^{ax} = \frac{x^r}{f^r(a)} e^{ax} \quad \ldots\ldots (v)$$

**WORKING RULE** To find the particular integral $\frac{1}{f(D)} e^{ax}$ we replace D by $a$, provided $f(a) \neq 0$. In case $f(a) = 0$, we multiply the numerator by x and differentiate the denominator w.r.t. D and apply the rule, provided $f'(a) \neq 0$ and so on.

**NOTE** In order to write the P.I. of $\sinh ax$ or $\cosh ax$, express them in terms of their exponential equivalents $\frac{1}{2}(e^{ax} - e^{-ax})$ and $\frac{1}{2}(e^{ax} + e^{-ax})$ respectively. Similarly, we can proceed for $\sin ax$ and $\cos ax$ by converting them to their respective exponential equivalents.

**EXAMPLE 1** Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4 + 3e^{5x}$.

**SOLUTION** The auxiliary equation is

$$D^2 - 3D + 2 = 0$$

or $(D - 2)(D - 1) = 0$

$$\Rightarrow \quad D = 2, 1.$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(4 + 3e^{5x})$$

$$\frac{1}{D^2 - 3D + 2} 4e^{0x} + \frac{3}{D^2 - 3D + 2} e^{5x}$$

$$= \frac{4}{(0)^2 - 3(0) + 2} e^{0x} + \frac{3}{(5)^2 - 3(5) + 2} e^{5x}$$

$$= \frac{4}{2} + \frac{3}{12} e^{5x} = 2 + \frac{1}{4} e^{5x}.$$

Hence complete solution is

$$y = c_1 e^{2x} + c_2 e^{x} + 2 + \frac{1}{4} e^{5x}.$$

**EXAMPLE 2** Find the particular integral of $(D - 2)(D + 1)^2 y = e^{2x} + 2 \sinh x$.

**SOLUTION** $$\text{P.I.} = \frac{1}{(D - 2)(D + 1)^2} (e^{2x} + 2 \sinh x)$$

$$= \frac{1}{(D - 2)(D + 1)^2} (e^{2x} + e^{x} - e^{-x}) \quad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now

$$\frac{1}{(D - 2)(D + 1)^2} e^{2x} = \frac{1}{D - 2} \cdot \frac{1}{(2 + 1)^2} e^{2x} = \frac{1}{9} \cdot \frac{1}{D - 2} e^{2x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} \cdot e^{2x},$$

(since D – 2 becomes zero when D = 2, so being case of failure, we use $\frac{d}{dD}(D-2) = 1$, and pre-multiply by x)

$$= \frac{1}{9} xe^{2x}.$$

$$\frac{1}{(D-2)(D+1)^2} e^x = \frac{1}{1-2} \cdot \frac{1}{(1+1)^2} e^x = -\frac{1}{4} e^x$$

$$\frac{1}{(D-2)(D+1)^2} e^{-x} = \frac{1}{-1-2} \cdot \frac{1}{(D+1)^2} e^{-x} = -\frac{1}{3} \cdot \frac{1}{(D+1)^2} e^{-x}$$

$$= -\frac{1}{3} \cdot \frac{x^2}{\underline{2}} e^{-x} \qquad \text{(case of failure)}$$

Hence $\quad P.I. = \frac{1}{9} xe^{2x} - \frac{1}{4} e^x - \frac{1}{6} x^2 e^{-x}.$

**8.3.4 CASE II** When X is of the form sin $ax$ or cos $ax$ and f(D) contains only even powers of D, say f(D) = f($D^2$), then the P.I. is given by

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$$

and $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax,$ provided $f(-a^2) \neq 0.$

**PROOF** We have by successive differentiation

$$D \sin ax = a \cos ax$$

$$D^2 \sin ax = (-a^2) \sin ax$$

$$D^4 \sin ax = (-a^2)^2 \sin ax \Rightarrow (D^2)^2 \sin ax = (-a^2)^2 \sin ax$$

$$D^6 \sin ax = (-a^2)^3 \sin ax \Rightarrow (D^2)^3 \sin ax = (-a^2)^3 \sin ax$$

$$\ldots \quad \ldots \quad \ldots$$

$$(D^2)^n \sin ax = (-a^2)^n \sin ax$$

Hence $f(D^2) \sin ax = f(-a^2) \sin ax$

or $\frac{1}{f(D^2)} f(D^2) \sin ax = \frac{1}{f(D^2)} f(-a^2) \sin ax$

or $$\sin ax = f(-a^2)\frac{1}{f(D^2)}\sin ax$$

or $$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax, \text{ provided } f(-a^2) \neq 0.$$

Similarly, it can be shown that

$$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax, \text{ provided } f(-a^2) \neq 0.$$

**REMARK** Since $D^3 = D^2 \cdot D = -a^2D$

$$D^4 = D^2 \cdot D^2 = (-a^2)(-a^2) = a^4$$

$$D^5 = D^4 \cdot D = a^4 \cdot D \text{ and so on.}$$

Hence if f(D) contains odd powers of D also, i.e., f(D) is a function of D and $D^2$, then by substituting $-a^2$ for $D^2$, we see that f(D) shall be reduced to a linear factor of the form p+Dq or p–Dq, which can further be converted to $f(D^2) = p^2 - D^2q^2$ by multiplying above and below by the conjugate factor of this linear factor. Then the term $D^2$ so obtained in the denominator can be replaced by $-a^2$. To elaborate this procedure we proceed to break up f(D) into its even and odd parts, and write

$$\frac{1}{f(D)}\sin ax = \frac{1}{\varphi(D^2) + D\psi(D^2)}\sin ax$$

$$= \frac{1}{\varphi(-a^2) + D\psi(-a^2)}\sin ax$$

$$= \frac{1}{p + Dq}\sin ax, \text{ where } p = \varphi(-a^2) \text{ and } q = \psi(-a^2)$$

$$= (p - Dq)\frac{1}{(p + Dq)(p - Dq)}\sin ax$$

$$= (p - Dq)\cdot\frac{1}{p^2 - D^2q^2}\sin ax$$

$$= (p - Dq)\cdot\frac{1}{p^2 + a^2q^2}\sin ax$$

$$= \frac{1}{p^2 + a^2q^2}\cdot(p - Dq)\sin ax$$

$$= \frac{1}{p^2 + a^2q^2}(p\sin ax - a\,q\cos ax)$$

The same procedure can be used to treat with cos $ax$.

### 8.3.5 ALTERNATE METHOD

We know that

$e^{iax} = \cos ax + i \sin ax$ (Euler's Theorem)

Thus P.I. for $e^{iax}$ = P.I. for $(\cos ax + i \sin ax)$

After evaluating the L.H.S. by short methods – case I, we deduce that

Real part of P.I. for $e^{iax}$ = P.I. for $\cos ax$

Imaginary part of P.I. for $e^{iax}$ = P.I. for $\sin ax$.

### 8.3.6 Case of Failure for $\frac{1}{f(D^2)} \sin ax$ or $\frac{1}{f(D^2)} \cos ax$.

We have proved that

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \text{ provided } f(-a^2) \neq 0.$$

The method fails if $f(-a^2) = 0$, since $\frac{1}{f(-a^2)} \sin ax$ becomes infinite.

But $f(-a^2) = 0$ means that $(D^2 + a^2)$ is a factor of $f(D^2)$.

Let $f(D^2) = (D^2 + a^2)\, \varphi(D^2)$, where $\varphi(-a^2) \neq 0$.

Then, $$\frac{1}{f(D^2)} \sin ax = \frac{1}{(D^2 + a^2)\, \varphi(D^2)} \sin ax$$

$$= \frac{1}{(D^2 + a^2)\, \varphi(-a^2)} \sin ax, \text{ provided } \varphi(-a^2) \neq 0$$

$$= \frac{1}{\varphi(-a^2)} \cdot \frac{1}{(D + ia)(D - ia)} \sin ax$$

$$= \frac{1}{\varphi(-a^2)} \cdot \frac{1}{2ia} \left[ \frac{1}{D - ia} - \frac{1}{D + ia} \right] \sin ax, \quad \ldots\ldots \text{(i)}$$

(resolving into partial fractions.)

Now by theorem 8.2.3, we have

$$\frac{1}{D - ia} \sin ax = e^{iax} \int e^{-iax} \sin ax \, dx$$

$$= e^{iax} \int e^{-iax} \left( \frac{e^{iax} - e^{-iax}}{2i} \right) dx$$

$$= \frac{e^{iax}}{2i} \int (1 - e^{-2iax}) \, dx$$

$$= \frac{e^{iax}}{2i} \left[ x + \frac{e^{-2iax}}{2ia} \right]$$

Replacing i with –i, we get

$$\frac{1}{D + ia} \sin ax = -\frac{e^{-iax}}{2i} \left[ x - \frac{e^{2iax}}{2ia} \right]$$

Hence (i) becomes

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{\varphi(-a^2)} \cdot \frac{1}{2ia} \left[ x \left( \frac{e^{iax} + e^{-iax}}{2i} \right) - \left( \frac{e^{iax} - e^{-iax}}{4i^2 a} \right) \right]$$

$$= \frac{1}{\varphi(-a^2)} \cdot \frac{1}{2a} \left[ -x \left( \frac{e^{iax} + e^{-iax}}{2} \right) + \left( \frac{e^{iax} - e^{-iax}}{4ia} \right) \right]$$

$$= \frac{1}{\phi(-a^2)} \left[ -\frac{x \cos ax}{2a} + \frac{\sin ax}{4a^2} \right]$$

Since $(D^2 + a^2)$ is a factor of $f(D^2)$ i.e., $D = \pm ia$, therefore, the part of the complementary function corresponding to this factor is $(c_1 \cos ax + c_2 \sin ax)$. Hence the term $\frac{\sin ax}{4a^2 \varphi(-a^2)}$ of the particular integral can be absorbed in C.F.

Hence $$\frac{1}{f(D^2)} \sin ax = -\frac{x \cos ax}{2a \varphi(-a^2)}$$

or $$\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax.$$

Similarly, it can be shown that

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax.$$

**REMARKS**

(i) Easy way to remember these results:

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2}\int \cos ax \, dx = \frac{x}{2} \cdot \frac{\sin ax}{a}$$

$$\frac{1}{D^2 + a^2} \sin ax = \frac{x}{2}\int \sin ax \, dx = -\frac{x}{2} \cdot \frac{1}{a} \cos ax$$

(ii) Another rule (without proof) for cases of failure applicable in cases I and II both.

$$\text{P.I.} = x \cdot \frac{1}{\text{diff. coeff. of denom. w.r.t.D}}$$ $e^{ax}$ or $\cos ax$ or $\sin ax$ as the case may be. If the denominator becomes zero again, we repeat the rule and so on. (See working rule (v) of case I,8.3.3)

(iii) In case we are required to find the P.I. for $\sin^2 x$, $\cos^2 x$, $\sin^3 x$ and $\cos^3 x$ we should change them in terms of multiple angles by using the results:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x], \cos^3 x = \frac{1}{4}[3\cos x + \cos 3x].$$

Similarly, the products like $\sin \alpha \cos \beta$ should be expressed as sum or difference of either sines or cosines.

**EXAMPLE 1** Solve $(D^3 + 1)\, y = \cos 2x$

**SOLUTION** Characteristic equation is

$$D^3 + 1 = 0 \Rightarrow (D + 1)(D^2 - D + 1) = 0$$

$$\therefore \quad D = -1, \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \text{ C.F.} = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{and P.I.} = \frac{1}{D^3 + 1} \cos 2x = \frac{1}{D^2 \cdot D + 1} \cos 2x$$

$$= \frac{1}{-4D + 1} \cos 2x \qquad \text{(replacing } D^2 \text{ by } -2^2 = -4)$$

Multiplying numerator and denominator by (1 + 4D) to make the denominator a function of $D^2$, we get

$$\text{P.I.} = \frac{1+4D}{1-16D^2}\cos 2x = \frac{1}{1+64}(1+4D)\cos 2x$$

$$= \frac{1}{1+64}[\cos 2x + 4D(\cos 2x)]$$

$$= \frac{1}{65}[\cos 2x - 8\sin 2x]$$

Hence complete solution is

$$y = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}(\cos 2x - 8\sin 2x).$$

**EXAMPLE 2** Solve $\frac{d^2y}{dx^2} + 4y = e^{3x} + \sin 2x$

**SOLUTION** Characteristic equation is

$$D^2 + 4 = 0 \Rightarrow D = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

and $$\text{P.I.} = \frac{1}{D^2+4}e^{3x} + \frac{1}{D^2+4}\sin 2x$$

Now $$\frac{1}{D^2+4}e^{3x} = \frac{1}{9+4}e^{3x} = \frac{1}{13}e^{3x}$$

and for $\frac{1}{D^2+4}\sin 2x$, since $(D^2 + 4)$ becomes zero when $D^2$ is replaced by $-2^2 = -4$, therefore, it is a case of failure and we proceed as follows:

$$\frac{1}{D^2+4}\sin 2x = x \cdot \frac{1}{2D}(\sin 2x) \quad \left\{\because \frac{d}{dD}(D^2+4) = 2D\right\}$$

$$= \frac{x}{2}\cdot\left[-\frac{\cos 2x}{2}\right] \quad \left\{\because \frac{1}{D} \text{ means integration}\right\}$$

$$= -\frac{1}{4}x\cos 2x$$

**ALITER:** $\frac{1}{D^2+4}\sin 2x =$ imaginary part of $\frac{1}{D^2+4}e^{2ix}$

Since $D^2 + 4$ vanishes for $D = 2i$, so we factorise it to get

$$\frac{1}{(D-2i)(D+2i)} e^{2ix} = \frac{1}{4i} \cdot \frac{1}{D-2i} e^{2ix} = \frac{1}{4i} xe^{2ix}$$

$$= -\frac{i}{4} x (\cos 2x + i \sin 2x)$$

$$= \frac{1}{4} x \sin 2x - \frac{i}{4} x \cos 2x$$

$$= -\frac{1}{4} x \cos 2x \qquad \text{(imaginary part)}$$

$$\therefore \quad \text{P.I.} = \frac{1}{13} e^{3x} - \frac{1}{4} x \cos 2x.$$

Thus the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{13} e^{3x} - \frac{1}{4} x \cos 2x.$$

**EXAMPLE 3** Solve the initial-value problem

$$(D^2 + 2D + 1) y = \cos x, \quad y(0) = 0, \quad y'(0) = 0$$

**SOLUTION** The auxiliary equation is

$$D^2 + 2D + 1 = 0 \Rightarrow (D+1)^2 = 0 \Rightarrow D = -1, -1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x) e^{-x}$$

and

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1} \cos x$$

$$= \text{real part of } \frac{1}{D^2 + 2D + 1} e^{ix}$$

$$= \text{real part of } \frac{1}{-1 + 2i + 1} e^{ix}$$

$$= \text{real part of } \frac{1}{2i} e^{ix}$$

$$= \text{real part of } \frac{-i}{2} (\cos x + i \sin x)$$

$$= \frac{1}{2} \sin x$$

$\therefore$ General solution of given equation is

$$y = (c_1 + c_2x)\, e^x + \frac{1}{2}\sin x$$

$$\therefore \quad y' = c_2 e^x - (c_1 + c_2x)\, e^x + \frac{1}{2}\cos x$$

Now $\quad y(0) = 0 \Rightarrow 0 = c_1$

and $\quad y'(0) = 0 \Rightarrow 0 = c_2 - c_1 + \frac{1}{2} \quad$ i.e., $\quad c_2 = -\frac{1}{2}$

Hence the particular solution is

$$y = \left(0 - \frac{1}{2}x\right)e^x + \frac{1}{2}\sin x$$

$$= -\frac{1}{2}x\, e^x + \frac{1}{2}\sin x$$

**8.3.7 CASE III** When X is of the form $x^m$ (or a polynomial of degree m), where m is positive integer.

Procedure for evaluating P.I. $= \frac{1}{f(D)}x^m$.

From f(D) we take out the lowest degree term. The remaining factor is of the form $[1 \pm \varphi(D)]$. This factor when taken to the numerator can be written as $[1 \pm \varphi(D)]^{-1}$ and then expanded binomially in ascending powers of D upto the mth power. Then we evaluate the result by operating with each operator of the expansion on $x^m$ keeping in mind that the powers of D beyond m need not be retained because (m + 1)th and higher derivatives of $x^m$ are zero.

**EXAMPLE 1** Solve $(D^3 + 3D^2 + 2D)\, y = x^2 + e^x$

**SOLUTION** Characteristic equation is

$$D(D^2 + 3D + 2) = 0$$

or $\quad D(D + 1)(D + 2) = 0 \Rightarrow D = 0, -1, -2.$

$$\therefore \quad C.F. = c_1 e^{0x} + c_2 e^{-x} + c_3 e^{-2x}$$

$$= c_1 + c_2 e^{-x} + c_3 e^{-2x}$$

Also $\quad P.I. = \frac{1}{D^3 + 3D^2 + 2D}(x^2 + e^x)$

Now making use of the procedure given in case III, we have

$$\frac{1}{D^3 + 3D^2 + 2D} x^2 = \frac{1}{2D\left[1 + \frac{3}{2}D + \frac{1}{2}D^2\right]} x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3}{2}D + \frac{1}{2}D^2\right) + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)^2 - \cdots\right]$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D + \frac{7}{4}D^2 \ldots\right] x^2$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{7}{2}x\right]$$

and $$\frac{1}{D^3 + 3D^2 + 2D} e^{-x} = x \cdot \frac{1}{\frac{d}{dD}[D^3 + 3D^2 + 2D]} e^{-x}$$ (case of failure)

$$= x \cdot \frac{1}{3D^2 + 6D + 2} e^{-x}$$

$$= x \cdot \frac{1}{3(-1)^2 + 6(-1) + 2} e^{-x}$$

$$= -x e^{-x}$$

Hence the complete solution is

$$y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{7}{2}x\right) - x e^{-x}$$

**EXAMPLE 2** Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2$ subject to the conditions

$$y(0) = 1,\ y(1) = 1$$

**SOLUTION** The characteristic equation is

$$D^2 + 4D + 4 = 0$$

or $(D+2)^2 = 0 \Rightarrow D = -2, -2.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-2x}$

Also P.I. $= \dfrac{1}{(D+2)^2} x^2$

$$= \frac{1}{4}\left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 - 2\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{D}{2}\right)^2 \ldots\right] x^2$$

$$= \frac{1}{4}\left[1 - D + \frac{3}{4}D^2\right] x^2, \text{ neglecting } D^3 \text{ and higher powers of D.}$$

$$= \frac{1}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

Hence the general solution of given equation is

$$y = (c_1 + c_2 x)\, e^{-2x} + \frac{1}{4}\left(x^2 - 2x + \frac{3}{2}\right)$$

Now $y(0) = 1 \Rightarrow 1 = c_1 + \dfrac{3}{8}$ i.e., $c_1 = \dfrac{5}{8}$

and $y(1) = 1 \Rightarrow 1 = (c_1 + c_2)\, e^{-2} + \dfrac{1}{4}\left(1 - 2 + \dfrac{3}{2}\right)$

$$\Rightarrow \quad 1 = \frac{c_1 + c_2}{e^2} + \frac{1}{8}$$

$$\Rightarrow c_1 + c_2 = \frac{7}{8} e^2$$

$$\Rightarrow \quad c_2 = \frac{7}{8} e^2 - c_1 = \frac{7}{8} e^2 - \frac{5}{8} = \frac{1}{8}(7e^2 - 5)$$

Thus the required solution is

$$y = \left[\frac{5}{8} + \frac{x}{8}(7e^2 - 5)\right] e^{2x} + \frac{1}{8}(2x^2 - 4x + 3)$$

**8.3.8 CASE IV** When X is of the form $e^{ax}$ V, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$$

(It is called Exponential Shift Theorem for inverse operators)

**PROOF** If u is a function of x; then

$$D(e^{ax} u) = e^{ax} Du + ae^{ax} u$$

$$= e^{ax} (Du + au)$$

$$= e^{ax} (D + a) u$$

$$D^2(e^{ax} u) = D[e^{ax} (D + a)u]$$

$$= e^{ax} (D^2 + aD)u + ae^{ax} (D + a) u$$

$$= e^{ax} |D^2 + 2aD + a^2| u$$

$$= e^{ax} (D + a)^2 u$$

Continuing this process of successive differentiation,

$$D^n(e^{ax} u) = e^{ax} (D + a)^n u$$

Hence $f(D) (e^{ax} u) = e^{ax} f(D + a)u$, where f(D) is a polynomial operator of order n.

Operating on both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} |f(D) (e^{ax} u)| = \frac{1}{f(D)} |e^{ax} f(D + a)u|$$

or $$e^{ax} u = \frac{1}{f(D)} |e^{ax} f(D + a)u|$$

Let $f(D + a)u = V$ so that $u = \frac{1}{f(D + a)} V$.

Then $e^{ax} \frac{1}{f(D + a)} V = \frac{1}{f(D)} (e^{ax} V)$

i.e., $\frac{1}{f(D)}(e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V$

**WORKING RULE** Take out $e^{ax}$ and in f(D) replace every D by $(D + a)$ so that f(D) converts to $f(D + a)$. Then operate $\frac{1}{f(D+a)}$ upon V which will be one of the cases already discussed.

**NOTE** By the above method we can find $\frac{1}{f(D)} e^{ax}$ very easily when $f(a)$ is zero (case of failure for case I) and also the particular integral corresponding to sin $ax$ or cos ax by considering them as the imaginary and real parts of $e^{iax}$ respectively.

**EXAMPLE 1** Solve $\frac{d^2y}{dx^2} - y = e^x (x^2 + 1) + x \sin x$

**SOLUTION** Characteristic equation is

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$$

$$\text{Also} \quad y_p = \frac{1}{D^2 - 1} e^x (x^2 + 1) + \frac{1}{D^2 - 1} x \sin x$$

But $\frac{1}{D^2 - 1} e^x (x^2 + 1) = e^x \frac{1}{(D + 1)^2 - 1} (x^2 + 1)$

$$= e^x \frac{1}{D^2 + 2D} (x^2 + 1)$$

$$= e^x \cdot \frac{1}{2D} \left(1 + \frac{D}{2}\right)^{-1} (x^2 + 1)$$

$$= e^x \cdot \frac{1}{2D} \left(1 - \frac{D}{2} + \frac{D^2}{4}\right)(x^2 + 1), \text{ neglecting } D^3$$

and higher powers of D.

$$= e^x \cdot \frac{1}{2D} \left[x^2 + 1 - \frac{1}{2} D(x^2 + 1) + \frac{1}{4} D^2(x^2 + 1)\right]$$

$$= e^x \cdot \frac{1}{2D} \left[x^2 + 1 - x + \frac{1}{2}\right]$$

$$= e^x \cdot \frac{1}{2D}\left(x^2 - x + \frac{3}{2}\right)$$

$$= \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right)$$

and $\dfrac{1}{D^2-1}\, x \sin x$ = imaginary part of $\dfrac{1}{D^2-1}\, x\, e^{ix}$

$$= \text{imaginary part of } e^{ix}\frac{1}{(D+i)^2-1}x$$

$$= \text{imaginary part of } e^{ix}\frac{1}{D^2+2iD-2}x$$

$$= \text{imaginary part of } -\frac{1}{2}e^{ix}\left[1-\left(iD+\frac{D^2}{2}\right)\right]^{-1}x$$

$$= \text{imaginary part of } -\frac{1}{2}e^{ix}\left[1+\left(iD+\frac{D^2}{2}\right)+\cdots\right]x$$

$$= \text{imaginary part of } -\frac{1}{2}e^{ix}[1+iD]x, \text{ ignoring } D^2 \text{ etc.}$$

$$= \text{imaginary part of } -\frac{1}{2}(\cos x + i \sin x)\,[x+i]$$

$$= -\frac{1}{2}\cos x - \frac{1}{2}x \sin x$$

Hence the general solution of given equation is

$$y = y_c + y_p$$

$$= c_1e^x + c_2e^{-x} + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right) - \frac{1}{2}(\cos x + x \sin x)$$

**EXAMPLE 2** Solve $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

**SOLUTION** The symbolic form of the equation is

$$(D^2+1)y = \operatorname{cosec} x$$

The characteristic equation

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$\therefore \quad y_c = c_1 \cos x + c_2 \sin x$$

Also
$$y_p = \frac{1}{D^2 + 1} \operatorname{cosec} x$$

$$= \frac{1}{(D - i)(D + i)} \operatorname{cosec} x$$

$$= \frac{1}{2i}\left[\frac{1}{D - i} - \frac{1}{D + i}\right] \operatorname{cosec} x$$

Now $\dfrac{1}{D - i} \operatorname{cosec} x = \dfrac{1}{D - i} \cdot \dfrac{1}{\sin x}$

$$= \frac{1}{D - i} e^{ix}\left(\frac{e^{-ix}}{\sin x}\right)$$

$$= e^{ix} \frac{1}{D + i - i} \cdot \frac{e^{-ix}}{\sin x}$$

$$= e^{ix} \frac{1}{D}\left(\frac{\cos x - i \sin x}{\sin x}\right)$$

$$= e^{ix} \frac{1}{D} (\cot x - i)$$

$$= e^{ix} (\log \sin x - ix)$$

Replacing i by –i, we get

$$\frac{1}{D + i} \operatorname{cosec} x = e^{-ix} (\log \sin x + ix)$$

$$\therefore \quad y_p = \frac{1}{2i}\left[e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix)\right]$$

$$= \log \sin x \left[\frac{e^{ix} - e^{-ix}}{2i}\right] - \frac{ix}{i}\left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

$$= \sin x \log \sin x - x \cos x$$

Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x$$

## EXERCISE 8.3

Solve

1. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

2. $\frac{d^2y}{dx^2} - y = \frac{1}{2}(e^x + e^{-x}) + 2\sin x$

3. $\frac{d^3y}{dx^3} + y = 5 + 3e^x - \cos^2\frac{1}{2}x$

4. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = e^{2x} + 3e^x$

5. $(D^3 + D^2 - D - 1)y = \sin 2x$

6. $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

7. $(D^2 - 5D + 6)y = x(x + e^x)$

8. $(D^3 - 3D + 2)y = e^{-2x} + 2\sinh x$

9. $(D^2 + 1)y = \sin x \sin 2x$

10. $(D^2 - 4)y = x\sinh x$

11. $(D^2 + 4D - 12)y = (x - 1)e^{2x}$

12. $(D^3 - D^2 - 6D)y = x^2 + 1$

13. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$

14. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = \cos x$

15. $\frac{d^2y}{dx^2} + 4y = e^x + \sin^2 x$

16. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8(x^2 + e^{2x} + \sin 2x)$

17. $\frac{d^2y}{dx^2} + y = 2e^x + x^3 + 5e^x \sin x + 6 + \cos x$

18. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 4y = xe^{-2x}$

19. $(D^2 - 6D + 13)y = e^{3x}\sin 4x + 2^x$

20. $(D^2 - 2D + 1)y = xe^x \sin x$

21. $(D^2 + 1)y = 6\cos^2 x$ subject to $y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0$

Solve the initial-value problems:-

22. $(D^2 + 1)y = 3x^2 - 4\sin x,\ y(0) = 0,\ y'(0) = 1$

23. $(D^2 + 5D + 6)y = e^{2x},\ y(0) = 0,\ y'(0) = 1$

24. $(D^2 - 4D + 3)y = 9x^2 + 4,\ y(0) = 6,\ y'(0) = 8$

25. $(D^2 + 1)y = x\sin x,\ y(0) = 1,\ y'(0) = 2$

26. $(D^2 - 2D - 3)y = 2e^{2x} + 10\sin 3x,\ y(0) = 2,\ y'(0) = 4$

## 8.4 LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS REDUCIBLE TO EQUATIONS WITH CONSTANT COEFFICIENTS

We will now study two types of linear differential equations with variable coefficients which can be transformed into linear differential equations with constant coefficients by making use of appropriate substitutions.

### 8.4.1 THE *CAUCHY–EULER EQUATION

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X, \quad \ldots\ldots \text{(i)}$$

where a's are constants and X is either a constant or a function of x, is known as **Cauchy–Euler** equation. It may be noted that the index of x and the order of derivative is same in each term of such an equation.

In order to transform (i) to a linear differential equation with constant coefficients we change the independent variable x to t by effecting the substitution

$$x = e^t \text{ i.e., } t = \log x \text{ so that } \frac{dt}{dx} = \frac{1}{x}$$

By the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\therefore \quad x \frac{dy}{dx} = \frac{dy}{dt} = \Delta y \quad \text{where } \Delta \equiv \frac{d}{dt}$$

$$\text{Also } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

* Augustin Louis Cauchy (1789—1857) —A French mathematician often called 'the father of modern analysis'. He wrote nearly 800 papers in mathematics, physics and astronomy. He was responsible for developing the concept of convergence of an infinite series and the theory of functions of a complex variable.

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\cdot\frac{d^2y}{dt^2}\cdot\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}(\Delta^2 - \Delta)y$$

or $$x^2\frac{d^2y}{dx^2} = \Delta(\Delta - 1)y$$

Again $$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d}{dx}\left[\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right]$$

$$= \frac{1}{x^2}\cdot\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) + \frac{d^2y}{dt^2}\cdot\frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{1}{x^2}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\cdot\frac{d}{dx}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\cdot\frac{dt}{dx} - \frac{2}{x^3}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx} + \frac{2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^2}\frac{d^3y}{dt^3}\cdot\frac{1}{x} - \frac{2}{x^3}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{d^2y}{dt^2}\cdot\frac{1}{x} + \frac{2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^3}\cdot\frac{d^3y}{dt^3} - \frac{3}{x^3}\cdot\frac{d^2y}{dt^2} + \frac{2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^3}\left[\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right]$$

or $$x^3\frac{d^3y}{dx^3} = (\Delta^3 - 3\Delta^2 + 2\Delta)y$$

$$= \Delta(\Delta - 1)(\Delta - 2)y$$

In general, we can show that

$$x^n\frac{d^ny}{dx^n} = \Delta(\Delta - 1)(\Delta - 2)\ldots(\Delta - n + 1)y$$

By substituting the values of $x\frac{dy}{dx}, x^2\frac{d^2y}{dx^2}, \cdots$, in (i) it gets transformed into a linear differential equation with constant coefficients having t as independent variable. This new equation can be solved by the methods already discussed. After getting solution in terms of y and t, we replace t by log x or $e^{mt}$ by $x^m$ in order to express our solution in terms of the original variables y and x.

**EXAMPLE 1** Solve $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

**SOLUTION** Let $x = e^t$ so that $t = \log x$ and $\frac{d}{dt} = \Delta$, say.

Then the given equation becomes

$$[\Delta(\Delta - 1) - 2]y = e^{2t} + e^{-t}$$

or $$(\Delta^2 - \Delta - 2)y = e^{2t} + e^{-t} \quad \ldots\ldots \text{(i)}$$

Its characteristic equation is

$$\Delta^2 - \Delta - 2 = 0$$

or $$(\Delta - 2)(\Delta + 1) = 0 \Rightarrow \Delta = 2, -1$$

$$\therefore \quad \text{C.F.} = c_1 e^{2t} + c_2 e^{-t}$$

$$\text{P.I.} = \frac{1}{(\Delta - 2)(\Delta + 1)} e^{2t} + \frac{1}{(\Delta - 2)(\Delta + 1)} e^{-t}$$

$$= \frac{1}{(\Delta - 2)(2 + 1)} e^{2t} + \frac{1}{(-1 - 2)(\Delta + 1)} e^{-t}$$

$$= \frac{1}{3}\frac{1}{\Delta - 2} e^{2t} - \frac{1}{3}\frac{1}{\Delta + 1} e^{-t}$$

$$= \frac{1}{3} e^{2t}\frac{1}{(\Delta + 2) - 2} \cdot 1 - \frac{1}{3} e^{-t}\frac{1}{(\Delta - 1) + 1} \cdot 1$$

$$= \frac{1}{3} e^{2t} \cdot \frac{1}{D}(1) - \frac{1}{3} e^{-t}\frac{1}{D}(1)$$

$$= \frac{1}{3} t e^{2t} - \frac{1}{3} t e^{-t}$$

Hence the general solution of (i) is

$$y = c_1 e^{2t} + c_2 e^{t} + \frac{1}{3} t e^{2t} - \frac{1}{3} t e^{-t}$$

and converting t to x, we get

$$y = c_1 x^2 + \frac{c^2}{x} + \frac{1}{3} x^2 \log x - \frac{1}{3x} \log x$$

which is the general solution of given equation.

**EXAMPLE 2** Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$

**SOLUTION** Let $x = e^t$ so that $t = \log x$

Then the given equation becomes

$$[\Delta(\Delta - 1) - 3\Delta + 5]y = e^{2t} \sin t \quad , \quad \text{where } \Delta = \frac{d}{dt}$$

or $(\Delta^2 - 4\Delta + 5)y = e^{2t} \sin t$

which is a linear equation with constant coefficients. Its characteristic equation

$$\Delta^2 - 4\Delta + 5 = 0 \Rightarrow \Delta = 2 \pm i$$

$$\therefore \text{C.F.} = e^{2t}(c_1 \cos t + c_2 \sin t)$$

$$\text{P.I.} = \frac{1}{\Delta^2 - 4\Delta + 5} e^{2t} \sin t$$

$$= e^{2t} \frac{1}{(\Delta + 2)^2 - 4(\Delta + 2) + 5} \sin t$$

$$= e^{2t} \frac{1}{\Delta^2 + 1} \sin t \quad \text{(case of failure)}$$

$$= -\frac{1}{2} t e^{2t} \cos t$$

Hence the general solution is

$$y = e^{2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{2} t e^{2t} \cos t$$

$$= x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{1}{2} x^2 \log x \cos(\log x)$$

**EXAMPLE 3** Solve the initial-value problem

$$\frac{d^2y}{dx^2} - \frac{6}{x^2}\,y = \frac{1}{x^2}\log x \quad ; \quad y(1) = \frac{1}{6}\,,\ y'(1) = -\frac{1}{6}$$

**SOLUTION** The given equation can be written as

$$x^2\frac{d^2y}{dx^2} - 6y = \log x$$

Let $x = e^t$ so that $t = \log x$ and $\frac{d}{dt} = \Delta$

Then the given equation reduces to

$$[\Delta(\Delta - 1) - 6]\,y = t$$

or $$(\Delta^2 - \Delta - 6)\,y = t \qquad \ldots\ldots \text{(i)}$$

which is a linear equation with constant coefficients.

The characteristics equation of (i) is

$$\Delta^2 - \Delta - 6 = 0$$

or $$(\Delta - 3)(\Delta + 2) = 0 \Rightarrow \Delta = 3, -2.$$

$$\therefore \text{C.F.} = c_1 e^{3t} + c_2 e^{-2t}$$

$$\text{P.I.} = \frac{1}{\Delta^2 - \Delta - 6}\,t$$

$$= -\frac{1}{6}\left(1 + \frac{\Delta - \Delta^2}{6}\right)^{-1} \cdot t$$

$$= -\frac{1}{6}\left[1 - \frac{\Delta}{6}\right]t = -\frac{1}{6}t + \frac{1}{6}$$

Thus the solution of (i) is

$$y = c_1 e^{3t} + c_2 e^{-2t} - \frac{1}{6}\left(t - \frac{1}{6}\right)$$

and that of given equation is

$$y = c_1 x^3 + c_2 x^{-2} - \frac{1}{6}\left(\log x - \frac{1}{6}\right) \quad \ldots\ldots \text{(ii)}$$

$$\therefore\ y' = 3c_1 x^2 - 2c_2 x^{-3} - \frac{1}{6x}$$

Now $y(1) = \frac{1}{6} \Rightarrow \frac{1}{6} = c_1 + c_2 - \frac{1}{6}\left(-\frac{1}{6}\right) \Rightarrow c_1 + c_2 = \frac{5}{36}$

and $y'(1) = -\frac{1}{6} \Rightarrow -\frac{1}{6} = 3c_1 - 2c_2 - \frac{1}{6} \Rightarrow 3c_1 - 2c_2 = 0$

Solving these two equations, we get $c_1 = \frac{1}{18}$, $c_2 = \frac{1}{12}$

Substituting in (ii) we get $y = \frac{1}{18}x^3 + \frac{1}{12}x^{-2} - \frac{1}{6}\left(\log x - \frac{1}{6}\right)$

or $y = \frac{1}{6}\left[\frac{x^3}{3} + \frac{1}{2x^2} - \log x + \frac{1}{6}\right]$

## 8.4.2 II. LEGENDRE'S LINEAR EQUATION

An equation of the form

$$P_0 (ax+b)^n \frac{d^n y}{dx^n} + P_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} (ax+b) \frac{dy}{dx} + P_n y = X \ldots \text{(i)}$$

where $P_0, P_1, P_2, \cdots, P_n$ are constants and X is a function of x, is called Legendre's Linear equation. It becomes Cauchy–Euler equation if a = 1 and b = 0.

Such an equation can be reduced to linear equation with constant coefficients by making the substitution

$$ax + b = e^t \text{ so that } t = \log(ax+b) \Rightarrow \frac{dt}{dx} = \frac{a}{ax+b}$$

Now $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dt}$$

or $(ax+b) \cdot \frac{dy}{dx} = a\Delta y$, where $\Delta = \frac{d}{dt}$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b} \cdot \frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\cdot\frac{dy}{dt} + \frac{a}{ax+b}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \frac{-a^2}{(ax+b)^2}\cdot\frac{dy}{dt} + \frac{a}{ax+b}\cdot\frac{d^2y}{dt^2}\cdot\frac{a}{(ax+b)}$$

$$= \frac{a^2}{(ax+b)^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

or $(ax+b)^2\frac{d^2y}{dx^2} = a^2(\Delta^2 - \Delta)y$

$$= a^2\Delta(\Delta-1)y$$

Similarly, we can show that

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3\Delta(\Delta-1)(\Delta-2)y$$

$$(ax+b)^n\frac{d^ny}{dx^n} = a^n\Delta(\Delta-1)(\Delta-2)\cdots(\Delta-n+1)y$$

On substituting these values of $(ax+b)\frac{dy}{dx}$, $(ax+b)^2\frac{d^2y}{dx^2}$, etc., our equation (i) is transformed into a linear differential equation with constant coefficients and hence can be solved by previous methods.

**EXAMPLE** Solve $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$.

**SOLUTION** Let $2x+1 = e^t$ so that $t = \log(2x+1)$, $\frac{dt}{dx} = \frac{2}{2x+1}$.

Then $(2x+1)\frac{dy}{dx} = 2\Delta y$, where $\Delta = \frac{d}{dt}$

and $(2x+1)^2\frac{d^2y}{dx^2} = 2^2\Delta(\Delta-1)y = 4\Delta(\Delta-1)y$.

Thus the given equation reduces to

$$[4\Delta(\Delta-1) - 12\Delta + 16]y = 8e^{2t}$$

or $$(\Delta^2 - 4\Delta + 4)y = 2e^{2t} \quad \ldots\ldots \text{(i)}$$

which is a linear equation with constant coefficients. Its characteristic equation

$$\Delta^2 - 4\Delta + 4 = 0 \Rightarrow \Delta = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 t)\, e^{2t}$$

$$\text{and} \quad \text{P.I.} = \frac{1}{\Delta^2 - 4\Delta + 4} 2e^{2t}$$

$$= 2e^{2t} \frac{1}{(\Delta + 2)^2 - 4(\Delta + 2) + 4} \cdot 1$$

$$= 2e^{2t} \frac{1}{\Delta^2} \cdot 1$$

$$= 2e^{2t} \frac{1}{\Delta} \cdot t$$

$$= 2e^{2t} \left(\frac{t^2}{2}\right)$$

$$= t^2 e^{2t}$$

Hence the complete solution of (i) is

$$y = (c_1 + c_2 t)\, e^{2t} + t^2 e^{2t}$$

Replacing $e^t$ by $(2x + 1)$ and t by $\log (2x + 1)$ we obtain

$$y = [c_1 + c_2 \log (2x + 1)]\, (2x + 1)^2 + [\log (2x + 1)]^2\, (2x + 1)^2$$

which is the required solution.

## EXERCISE 8.4

Solve

1. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x$

2. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = \frac{1}{x^2}$

3. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x + 1)^2$

4. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4 + 2 \log x$

5. $(x + 1)^2 \frac{d^2y}{dx^2} - 3(x + 1) \frac{dy}{dx} + 4y = x^2$

6. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x \log x$

7. $x^3\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} + xy = \sin(\log x)$

8. $x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x$

9. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

10. $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$

11. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

12. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = \sin 2[\log(x+1)]$

13. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

14. $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$ when $y(1) = 1$ and $y'(1) = -6$

15. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \log x$

subject to the conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 1$.

## 8.5 SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

The standard form of a linear differential equation of the second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \ldots\ldots \text{(i)}$$

where P, Q, R are functions of x or constants. We have already dealt with equations with constant coefficients and Cauchy-type homogeneous equations. In this section we intend to study some other types of second order equations which require special methods for their solution. In what follows we shall be concerned mainly with the reduction of equations of second order to those of the first order.

### 8.5.1 METHOD I CHANGE OF DEPENDENT VARIABLE (Solution from a known integral or solution by reduction of order)

Let $y = z$ be a known integral belonging to the complementary function of (i) so that

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} + Qz = 0 \quad \ldots\ldots \text{(ii)}$$

In order to solve (i) let us make the subsitution

$$y = vz, \quad \text{where } v = v(x) \quad \text{and} \quad z = z(x).$$

Then $\frac{dy}{dx} = v\frac{dz}{dx} + z\frac{dv}{dx}$

and $\frac{d^2y}{dx^2} = v\frac{d^2z}{dx^2} + 2\frac{dz}{dx}\cdot\frac{dv}{dx} + z\frac{d^2v}{dx^2}$

By substituting the values and rearranging them, equation (i) takes the shape of,

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{z}\frac{dz}{dx}\right)\frac{dv}{dx} + \frac{1}{z}\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx} + Qz\right)v = \frac{R}{z} \quad \ldots\ldots \text{(iii)}$$

But, by (ii), the coefficient of v in (iii) is zero. Thus (iii) gives

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{z}\frac{dz}{dx}\right]\frac{dv}{dx} = \frac{R}{z} \quad \ldots\ldots \text{(iv)}$$

Now substituting p for $\frac{dv}{dx}$ and $\frac{dp}{dx}$ for $\frac{d^2v}{dx^2}$, we have

$$\frac{dp}{dx} + \left(P + \frac{2}{z}\cdot\frac{dz}{dx}\right)p = \frac{R}{z} \quad \ldots\ldots \text{(v)}$$

which is a linear equation in p and x. Hence we see that (i) has been reduced to a linear equation of first order and can be solved for p by methods already considered. From $p\left(\text{i.e., } \frac{dv}{dx}\right)$, by integrating we shall determine v. Substituting this value of v in the assumed solution $y = vz$ we shall obtain the complete solution of the given equation.

**NOTE** In addition to finding the C.F. and then choosing an integral out of infinite solutions corresponding to arbitrary constants involved, we state some rules which will be helpful in finding an integral belonging to the complementary function.

(i) $y = e^{ax}$ is a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ if $a^2 + Pa + Q = 0$.
In particular, when $a = 1$, $y = e^x$ is a part of C.F. if $1 + P + Q = 0$.
When $a = -1$, $y = e^{-x}$ is a part of C.F. if $1 - P + Q = 0$.

(ii) $y = x^m$ is a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ if $m(m-1) + Pmx + Qx^2 = 0$.
In particular, when $m = 1$, $y = x$ is a part of the C.F. if $P + Qx = 0$.
When $m = 2$, $y = x^2$ is a part of the C.F. if $2 + 2Px + Qx^2 = 0$.

**REMARK** The method used for solving equation (i) is applicable to obtain from any linear differential equation a new equation of reduced order.

**EXAMPLE 1** Solve $\frac{d^2y}{dx^2} + y = \sec^3 x$

**SOLUTION** The solution of $\frac{d^2y}{dx^2} + y = 0$ is

$$y_c = c_1 \sin x + c_2 \cos x$$

Taking $c_1 = 0$ and $c_2 = 1$ we see that $y = \cos x$ is a part of the C.F. So we assume that

$$y = v \cos x$$

is a particular solution of the given equation. Then

$$\frac{dy}{dx} = v(-\sin x) + \cos x \frac{dv}{dx} = -v \sin x + \cos x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -v \cos x - \sin x \frac{dv}{dx} + (-\sin x) \frac{dv}{dx} + \cos x \frac{d^2v}{dx^2}$$

$$= -v \cos x - 2 \sin x \frac{dv}{dx} + \cos x \frac{d^2v}{dx^2}$$

Substitution in the given equation yields

$$-v \cos x - 2 \sin x \frac{dv}{dx} + \cos x \frac{d^2v}{dx^2} + v \cos x = \sec^3 x$$

or $$\cos x \frac{d^2v}{dx^2} - 2 \sin x \frac{dv}{dx} = \sec^3 x$$

or $$\frac{d^2v}{dx^2} - 2 \tan x \frac{dv}{dx} = \sec^4 x$$

Let $p = \frac{dv}{dx}$ so that $\frac{dp}{dx} = \frac{d^2v}{dx^2}$

Then the above equation becomes

$$\frac{dp}{dx} - 2 \tan x \cdot p = \sec^4 x$$

which is a linear equation of the first order.

Its I.F. $= e^{-2\int \tan x\,dx} = e^{2\log\cos x} = e^{\log\cos^2 x} = \cos^2 x.$

Thus $p\cos^2 x = \int \cos^2 x \cdot \sec^4 x\,dx = \int \sec^2 x\,dx = \tan x.$

or $p = \dfrac{\tan x}{\cos^2 x} \Rightarrow \dfrac{dv}{dx} = \dfrac{\sin x}{\cos^3 x}$

$\therefore$ $v = \int \cos^{-3} x \sin x\,dx = \dfrac{1}{2}\sec^2 x$, the constant of integration is neglected because of our choice of particular solution.

Hence a particular solution of the given equation is

$$y_p = v\cos x = \frac{1}{2}\sec^2 x \cos x = \frac{1}{2}\sec x$$

and the general solution is

$$y = c_1 \sin x + c_2 \cos x + \frac{1}{2}\sec x$$

**NOTE**

The student is advised to solve by choosing $c_1 = 1$ and $c_2 = 0$.

**EXAMPLE 2** Solve $x^2\dfrac{d^2y}{dx^2} - (x^2 + 2x)\dfrac{dy}{dx} + (x + 2)y = x^3e^x$ ..... (1)

**SOLUTION** Converting equation (1) into the standard form, we have

$$\frac{d^2y}{dx^2} - \left(1 + \frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = x e^x \qquad \text{..... (2)}$$

Since $P = -1 - \dfrac{2}{x}$ and $Q = \dfrac{1}{x} + \dfrac{2}{x^2}$, so

$$P + Qx = -1 - \frac{2}{x} + \left(\frac{1}{x} + \frac{2}{x^2}\right)x = 0$$

$\therefore$ $y = x$ is an integral of $y'' + Py' + Qy = 0$.

Let $y = vx$ be the complete solution. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \qquad \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

and hence equation (2) becomes

$$2\frac{dv}{dx} + x\frac{d^2v}{dx^2} - \left(1 + \frac{2}{x}\right)\left(v + x\frac{dv}{dx}\right) + \left(\frac{1}{x} + \frac{2}{x^2}\right)vx = x e^x$$

or $x\frac{d^2v}{dx^2} - x\frac{dv}{dx} = x e^x$

or $\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$ ..... (3)

Let $p = \frac{dv}{dx}$ so that $\frac{dp}{dx} = \frac{d^2v}{dx^2}$. Then (3) gives

$\frac{dp}{dx} - p = e^x$, which is a linear equation.

Its I.F. $= e^{-\int dx} = e^{-x}$

and its solution is

$p e^{-x} = \int e^{-x} \cdot e^x dx + c_1 = x + c_1$

$\therefore \quad p = \frac{dv}{dx} = x e^x + c_1 e^x \quad \Rightarrow dv = (x e^x + c_1 e^x) dx$

Integrating, we have

$v = x e^x - \int 1 \cdot e^x dx + c_1 e^x + c_2$

$= (x - 1)e^x + c_1 e^x + c_2$

Therefore, the general solution is given by

$y = vx$

$= x(x - 1)e^x + c_1 x e^x + c_2 x$

**EXAMPLE 3** Solve $x\frac{d^2y}{dx^2} - (2x + 1)\frac{dy}{dx} + (x + 1)y = (x^2 + x - 1)e^x$ ... (1)

**SOLUTION** To express in the standard form $y'' + Py' + Qy = R$ we divide (1) by x and get

$\frac{d^2y}{dx^2} - \left(2 + \frac{1}{x}\right)\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \left(x + 1 - \frac{1}{x}\right)e^x$

Here $P = -2 - \frac{1}{x}$, $Q = 1 + \frac{1}{x}$

$\therefore \quad 1 + P + Q = 1 - 2 - \frac{1}{x} + 1 + \frac{1}{x} = 0$

Hence $y = e^x$ is a particular integral of the given equation with its right member replaced by zero.

Let $y = ve^x$ be the general solution of (1). Then

$$\frac{dy}{dx} = ve^x + e^x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}$$

Hence the given equation becomes

$$x\left(ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}\right) - (2x+1)\left(ve^x + e^x \frac{dv}{dx}\right) + (x+1)\,ve^x = (x^2 + x - 1)e^x$$

Dividing throughout by $e^x$ and simplifying, we get

$$x\frac{d^2v}{dx^2} - \frac{dv}{dx} = x^2 + x - 1$$

or $$\frac{d^2v}{dx^2} - \frac{1}{x} \cdot \frac{dv}{dx} = x + 1 - \frac{1}{x}$$

Taking $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, we have

$$\frac{dp}{dx} - \frac{1}{x}\,p = x + 1 - \frac{1}{x}$$

This being a linear equation of the first order in p, its

$$\text{I.F.} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

and its solution is

$$p \cdot \frac{1}{x} = \int \frac{1}{x}\left(x + 1 - \frac{1}{x}\right)dx = x + \log x + \frac{1}{x} + c_1$$

$$\therefore \quad \frac{dv}{dx} = x^2 + x \log x + 1 + c_1 x$$

Integrating, we obtain

$$v = \frac{x^3}{3} + \left[\frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx\right] + x + c_1 \frac{x^2}{2} + c_2$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + x + \frac{1}{2}c_1x^2 + c_2$$

Hence the required solution is

$$y = ve^x = e^x\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + x + \frac{1}{2}c_1 x^2 + c_2\right]$$

## EXERCISE 8.5

Solve

1. $\frac{d^2y}{dx^2} + y = \text{cosec } x$

2. $\sin^2 x \frac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution.

3. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3$

4. $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$

5. $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

6. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + (1-x)y = x^2e^{-x}$

7. $x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^{2x}$

8. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$, given that $y = \sin x$ is a solution.

9. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2}$

10. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, given that $y = x + \frac{1}{x}$ is one integral.

11. $(x^2D^2 - 3xD + 3)y = 2x^3 - x^2$, where $D \equiv \frac{d}{dx}$

12. $(x-2)\frac{d^2y}{dx^2} - (4x-7)\frac{dy}{dx} + (4x-6)y = 0$

13. $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

14. $(x-1)\left(\frac{d^2y}{dx^2} - x + 1\right) - x\frac{dy}{dx} + y = 0$

15. $(1+x)\frac{d^2y}{dx^2} + (4x+5)\frac{dy}{dx} + (4x+6)y = e^{-2x}$

## 8.6 METHOD II. VARIATION OF PARAMETERS

This method is used to find a solution of the general linear differential equation when the solution of the corresponding homogeneous equation is known. But we shall restrict its illustration to determining the complete primitive of a second order linear equation whose complementary function is known. The method of variation of parameters is inferior in practical value to that of reduction of order, as it requires a complete knowledge of the complementary function instead of only one part of it. Moreover, its application to third or higher order equations involves too much labour.

Consider the second order linear differential equation

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ where R is a function of x and coefficient of $\frac{d^2y}{dx^2}$ is unity.

Let u and v be linearly independent solutions of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \ldots\ldots \text{(ii)}$$

so that the complementary function of (i) is

$$y = c_1u + c_2v \quad \ldots\ldots \text{(iii)}$$

where $c_1$ and $c_2$ are arbitrary constants.

Replacing the arbitrary constants $c_1$ and $c_2$ in (iii) with unknown functions of x, say A and B respectively, we get

$$y = Au + Bv \quad \ldots\ldots \text{(iv)}$$

where the functions A(x) and B(x) are so chosen that (iv) is a solution of (i). To determine two unknowns A and B we require to impose two conditions on them. We have already imposed one condition that the assumed solution (iv) satisfies (i). We are therefore at liberty to impose the second condition arbitrarily according to our convenience and to our advantage. With this in mind we proceed to differentiate (iv), and get

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} + u\frac{dA}{dx} + v\frac{dB}{dx} \quad \ldots\ldots \text{(v)}$$

At this stage, instead of involving ourselves with derivatives of A and B of higher order than the first, for our convenience, as a second condition we choose A and B such that

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \quad \ldots\ldots \text{(vi)}$$

Then a simplified form of (v) is

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} \quad \ldots\ldots \text{(vii)}$$

Differentiating (vii) w.r.t. x, we obtain

$$\frac{d^2y}{dx^2} = A\frac{d^2u}{dx^2} + \frac{du}{dx}\cdot\frac{dA}{dx} + B\frac{d^2v}{dx^2} + \frac{dv}{dx}\cdot\frac{dB}{dx} \quad \ldots\ldots \text{(viii)}$$

Because $y = A(x)\,u + B(x)\,v$ is to be a solution of (i), we substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively from (iv), (v) and (viii) in equation (i), and get on re-arranging the terms

$$A\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + B\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right) + \frac{dA}{dx}\cdot\frac{du}{dx} + \frac{dB}{dx}\cdot\frac{dv}{dx} = R$$

or $A\cdot 0 + B\cdot 0 + \frac{dA}{dx}\cdot\frac{du}{dx} + \frac{dB}{dx}\cdot\frac{dv}{dx} = R$, since u and v are solutions of (ii).

or $$\frac{du}{dx}\cdot\frac{dA}{dx} + \frac{dv}{dx}\cdot\frac{dB}{dx} = R \quad \ldots\ldots \text{(ix)}$$

Solving (vi) and (ix) for $\frac{dA}{dx}$ and $\frac{dB}{dx}$, we have

$$\frac{dA}{dx} = \frac{-vR}{u\cdot\frac{dv}{dx} - \frac{du}{dx}\cdot v} \quad \text{and} \quad \frac{dB}{dx} = \frac{uR}{u\cdot\frac{dv}{dx} - \frac{du}{dx}\cdot v},$$

where the denominator

$$u\cdot\frac{dv}{dx} - \frac{du}{dx}\cdot v = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix}$$

is not zero as u and v are linearly independent.

Integrating these results we determine the values of A and B which are given by

$$A = \int \frac{-vR}{uv' - u'v}\,dx + c, \quad B = \int \frac{uR}{uv' - u'v}\,dx + d \qquad \ldots\ldots (x)$$

where c and d are arbitrary constants. Substitution for A and B in (iv) will yield the required y.

Note (i) Since the solution is obtained by varying or replacing the arbitrary constants present in the complementary function, the above method is known as the method of variation of parameters-discovered by **Lagrange.**

(ii) Sometimes, the values of A and B are found more conveniently by the direct application of the formulae obtained in (x). See solved example 4. Students are advised to have practice of both approaches.

(iii) It may also be observed that the values of A and B will be as given in (x) provided the coefficient of $\frac{d^2y}{dx^2}$ in (i) is unity. Moreover, instead of general solution, if we are seeking a particular solution $y_p$ first, we shall discard the constants of integration c and d in (x). Then $y = y_c + y_p$ will give the required general solution. See solved examples 2 and 4.

**NOTE** Since the solution is obtained by varying or replacing the arbitrary constants present in the complementary function, the above method is known as the method of variation of parameters-discovered by **Lagrange.**

**EXAMPLE 1** Solve $y'' + y = \tan x$ ..... (1)

**SOLUTION** The complementary function corresponding to (1) is

$$a \cos x + b \sin x$$

where *a* and b are arbitrary constants.

Let us replace *a* and b by functions A(x) and B(x) to be determined so that

$$y = A(x) \cos x + B(x) \sin x \qquad \ldots\ldots (2)$$

is a solution of differential equation (1).

Differentiating, we have

$$y' = -A(x) \sin x + B(x) \cos x + A'(x) \cos x + B'(x) \sin x \qquad \ldots\ldots (3)$$

If we differentiate further at this stage, it will give rise to some more terms. So we make use of our choice of imposing one other condition on A(x) and B(x), namely

$$A'(x)\cos x + B'(x)\sin x = 0 \qquad \ldots\ldots (4)$$

which simplifies (3) to give

$$y' = -A\sin x + B\cos x$$

Differentiating again, we have

$$y'' = -A\cos x - B\sin x - A'\sin x + B'\cos x \qquad \ldots\ldots (5)$$

Substituting the values of y and y'' from (2) and (5) in the given differential equation (1), we obtain

$$-A'\sin x + B'\cos x = \tan x \qquad \ldots\ldots (6)$$

Solving (4) and (6) for A' and B', we have

$$A'(x) = -\frac{\sin^2 x}{\cos x} \quad \text{and} \quad B'(x) = \sin x$$

or $$A'(x) = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x \quad \text{and} \quad B'(x) = \sin x$$

By integration, it gives

$$A(x) = \sin x - \log(\sec x + \tan x) + c_1$$

and $$B(x) = -\cos x + c_2$$

Substituting these values of A(x) and B(x) in (2), we get the required solution as

$$y = (c_1\cos x + c_2\sin x) - \cos x \log(\sec x + \tan x)$$

This can be verified by substitution in (1) as the general solution.

**EXAMPLE 2** Use the method of variation of parameters to find the particular solution and the general solution of

$$\frac{d^2y}{dx^2} + y = \tan x \sec x \qquad \ldots\ldots (1)$$

**SOLUTION** The characteristic equation is

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

Thus $$y_c = c_1\cos x + c_2\sin x$$

Let us assume that

$$y = A\cos x + B\sin x \quad \ldots\ldots (2)$$

is a particular solution of the given equation.

Then $y' = -A\sin x + B\cos x + A'\cos x + B'\sin x$.

Choosing $A'\cos x + B'\sin x = 0$, $\ldots\ldots (3)$

we have $y' = -A\sin x + B\cos x$

Thus $y'' = -A\cos x - B\sin x - A'\sin x + B'\cos x \quad \ldots\ldots (4)$

Substituting the values of y and $y''$ from (2) and (4) in (1), we obtain

$$-A'\sin x + B'\cos x = \tan x \sec x \quad \ldots\ldots (5)$$

Solving (3) and (5) for $A'$ and $B'$, we have

$$A' = -\sin x \tan x \sec x = -\tan^2 x \text{ and } B' = \tan x$$

On integration, we get

$$A = \int (1 - \sec^2 x)dx = x - \tan x \quad \ldots\ldots (6)$$

and $$B = \int \tan x\, dx = \log \sec x \quad \ldots\ldots (7)$$

Note that we have discarded the arbitrary constants in (6) and (7) because we are seeking only a particular solution.

Substituting the values of A and B in (2) the particular solution is given by

$$y_p = (x - \tan x)\cos x + \sin x \log \sec x.$$

$$= x\cos x - \sin x + \sin x \log \sec x$$

Hence the gneral solution of equation (1) is

$$y = y_c + y_p$$

or $$y = c_1 \cos x + c_3 \sin x + x\cos x + \sin x \log \sec x.$$

where the term $(-\sin x)$ of the particular integral has been absorbed in the complementary function.

**EXAMPLE 3** Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = e^x \tan 2x \quad \ldots\ldots (1)$

**SOLUTION** The characteristic equation is

$$D^2 - 2D + 5 = 0 \Rightarrow D = 1 \pm 2i$$

Thus $y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$ ..... (2)

Let $y_p = e^x (u_1 \cos 2x + u_2 \sin 2x)$

$= u_1 (e^x \cos 2x) + u_2 (e^x \sin 2x)$, ..... (3)

where $u_1$ and $u_2$ are unknown functions of x.

Differentiating (3) w.r.t.x, we get

$y_p' = u_1' (e^x \cos 2x) + u_1(e^x \cos 2x - 2 e^x \sin 2x) + u_2'(e^x \sin 2x) + u_2(e^x \sin 2x + 2e^x \cos 2x)$ .(4)

Setting $u_1' (e^x \cos 2x) + u_2' (e^x \sin 2x) = 0$

i.e., $u_1' \cos 2x + u_2' \sin 2x = 0$ ..... (5)

we have from equation (4),

$y_p' = u_1 (e^x \cos 2x - 2e^x \sin 2x) + u_2 (e^x \sin 2x + 2e^x \cos 2x)$

At this stage the normal procedure is to find $y_p''$ and then substitute for $y_p$, $y_p'$, $y_p''$ in the given equation. During the course of simplification the terms involving $u_1$ and $u_2$ cancel out. So we do not bother to find them and shall calculate only those terms in $y_p''$ which involve $u_1'$ and $u_2'$. Substituting the terms containing $u_1'$, $u_2'$ equation (1) gives

$u_1' (e^x \cos 2x - 2e^x \sin 2x) + u_2' (e^x \sin 2x + 2e^x \cos 2x) = e^x \tan 2x$

or $u_1' (\cos 2x - 2 \sin 2x) + u_2' (\sin 2x + 2\cos 2x) = \tan 2x$ ..... (6)

To solve (5) and (6) for $u_1'$ and $u_2'$ we first calculate:

$$\begin{vmatrix} \cos 2x & \sin 2x \\ \cos 2x - 2\sin 2x & \sin 2x + 2\cos 2x \end{vmatrix}$$

$= \cos 2x \sin 2x + 2\cos^2 2x - \sin 2x \cos 2x + 2 \sin^2 2x$

$= 2 (\cos^2 2x + \sin^2 2x)$

$= 2 \neq 0$

$$\therefore \quad u_1' = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ \tan 2x & \sin 2x + 2\cos 2x \end{vmatrix} = -\frac{1}{2}\tan 2x \sin 2x$$

$$u_2' = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ \cos 2x - 2\sin 2x & \tan 2x \end{vmatrix} = \frac{1}{2}\cos 2x \tan 2x = \frac{1}{2}\sin 2x$$

Hence $u_1 = -\frac{1}{2}\int \tan 2x \sin 2x \, dx$ and $u_2 = \frac{1}{2}\int \sin 2x \, dx$.

Substituting these values of $u_1$ and $u_2$ in (3) we get the required solution.

**Example 4.** Solve $y'' + y = \dfrac{1}{1 + \sin x}$

**Solution.** As in solved example 2,

$$y_c = c_1 \cos x + c_2 \sin x$$

Suppose that $y_p = A \cos x + B \sin x$

To find the unknown functions A and B let us apply the formulae (x) directly. Here

$$u = \cos x, \quad v = \sin x, \quad R = \frac{1}{1 + \sin x}$$

Thus $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v = \cos^2 x + \sin^2 x = 1 \neq 0$

Therefore, by using formulae (x), we have

$$A = \int \frac{-vR}{uv' - u'v}\, dx = \int \frac{-\sin x}{1 + \sin x}\, dx$$

$$= -\int dx + \int \frac{dx}{1 + \sin x}$$

$$= -x + \tan x - \sec x$$

and $B = \int \dfrac{uR}{uv' - u'v}\, dx = \int \dfrac{\cos x}{1 + \sin x}\, dx = \log |(1 + \sin x)|$

Consequently, $y_p = A \cos x + B \sin x$

$$= -x \cos x + \sin x - 1 + \sin x \log |(1 + \sin x)|$$

Hence the general solution is

$$y = y_c + y_p$$

$$= c_1 \cos x + c_2 \sin x - x \cos x + \sin x - 1 + \sin x \log |(1 + \sin x)|$$

## EXERCISE 8.6

Use the method of variation of parameters to solve (primes denote derivatives with respect to x)

1. $y'' + y = \operatorname{cosec} x$

2. $y'' + 9y = \sec 3x$

3. $y'' + y = \tan^3 x$

4. $y'' + y = \dfrac{2}{1 + e^x}$

5. $y'' + y = \sec x \operatorname{cosec} x$

6. $y'' + y = \dfrac{1}{1 + \sin x}$

7. $y'' + 4y' + 5y = e^{-2x} \sec x$

8. $x^2 y'' + xy' - y = x^2 e^x$

9. $y'' - 2y' + y = e^x \sin^{-1} x$

10. $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

11. $y'' + 3y' + 2y = \dfrac{e^{-x}}{x}$

12. $y'' - 2y' + y = x e^x \log x, \; x > 0$

13. $(x - 1) y'' - xy' + y = (x - 1)^2$

14. Solve $x^2y'' - 2x(1 + x)y' + 2(1 + x)y = x^3$, given that the complementary function is $c_1x + c_2x e^{2x}$.

Use the formulae (x) directly, to solve

15. $(D^2 - 4D + 4)y = e^{2x} \sec^2 x$

16. $(D^2 + 2D + 5)y = 4e^{-x} \tan 2x$

17. $(D^2 - 4D + 4)y = \dfrac{e^{2x}}{1 + x}$

18. $(D^2 + 2D + 1)y = e^{-x} \log x$

## 8.6.1. METHOD III. UNDETERMINED COEFFICIENTS

Another useful and special method to find particular integral of a non-homogeneous linear differential equation with constant coefficients,

$$f(D)y = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)y = X(x), \quad \ldots (1)$$

is the method of undetermined coefficients. It is also known as the method of "educated guessing". The right hand side X includes a restricted class of functions like a polynomial, an exponential function, a sine or cosine or a finite product of two or more of such functions. Although restricted, but this class of functions includes functions of frequent accurrence and considerable importance in various physical applications. This method has one clear advantage over the method of variation of parameters that, if applicable, it is relatively simpler since it involves differentiation whereas method of variation of parameters involves integration which becomes intriguing at times.

We proceed by assuming for particular integral $y_p$ a trial solution of the form

$$y_p = Au_1(x) + Bu_2(x) + Cu_3(x) + \ldots + Ku_r(x), \quad \ldots (2)$$

consisting of a finite linear combination of functions $u_1(x), u_2(x), \ldots, u_r(x)$ which are the terms of X and those arising from these terms by differentiation and A, B, C, ...., K are constants to be determined such that (2) is a solution of (1).

For example,

(i) if $X = x^4$, we assume (2) to be $y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$.

(ii) if $X = e^x + e^{2x}$, we take (2) as $y_p = Ae^x + Be^{2x}$, because no new terms are obtained by differentiating $e^x$ and $e^{2x}$.

(iii) if $X = \sin ax$ or $\cos ax$, we try $y_p = A \sin ax + B \cos ax$

(iv) if $X = x^2 \sin x$, then the trial solution will be a linear combination of all products obtained by multiplying the various members of the set $\{x^2, x, 1\}$ arising from $x^2$ by the various members of the set $\{\sin x, \cos x\}$ arising from $\sin x$. Thus the form of the trial solution in this case is

$$y_p = (Ax^2 + Bx + C) \sin x + (Dx^2 + Ex + F) \cos x$$

(v) if $X = x^2 e^{3x} \cos 3x$, then we assume

$$y_p = (Ax^2 + Bx + C) e^{3x} \cos 4x + (Dx^2 + Ex + F) e^{3x} \sin 4x.$$

Thus the method of undetermined coefficients is applicable on (1) where X(x) is a linear combination of functions of the form $x^m e^{\alpha x} \cos \beta x$, $x^m e^{\alpha x} \sin \beta x$, where m is a non-negative integer and $\alpha, \beta$ are real numbers.

Once the form for a trial solution $y_p$ is determined, the actual $y_p$ is obtained by substituting the trial solution into the original differential equation (1), equating

coefficients of like terms on both sides of the resulting identity and solving the so obtained set of equations for undetermined coefficients A, B, C etc.

**Failure**

The method fails if $X = \sec x$, since the number of new terms obtained by differentiating sec x is infinite. In general, we cannot apply this method to differential equations in which X has $\ln |x|$, $\tan x$, variable coefficients, negative powers of x(like $x^{-1}$) or $x^{1/3} \cos x$ etc., since it is difficult to guess the form of the trial function in such cases.

**Modification**

(i) If a term contained in X is also a term of the complementary function obtained from the associated homogeneous equation $f(D)y = 0$ corresponding to (1), then the method fails. But, in such a case we can still succeed by introducing a factor of x or $x^2$ etc. into the proposed trial solution for $y_p$. The powers of x to be introduced are determined by the multiplicity of the roots of the characteristic equation. In fact, if any term of X, say u, is also a term of complementary function corresponding to a root m of multiplicity s, then we shall introduce a term $x^s u$ plus terms arising from it by differentiation. For example, when finding a particular integral of the differential equation

$$(D - 2)^2 (D + 3)y = e^{2x} + x^2,$$

the proposed solution is of the form

$$y_p = Ax^2 e^{2x} + Bx e^{2x} + Ce^{2x} + Dx^2 + Ex + F.$$

Here the first three terms arise from the fact that the term $e^{2x}$ of X is also a term of the complementary function corresponding to a double root $m = 2$. Hence we have made use of $x^s u = x^2 e^{2x}$ and all terms arising from it by differentiation.

(ii) If a term of X is $x^t u$ and u is a term of the complementary function corresponding to a root m of multiplicity s, then the trial function $y_p$ must contain the term $x^{t+s} u$ plus the terms arising from it by differentiation. For example, if we wish to find a particular integral of the differential equation

$$(D - 2)^3 (D + 3)y = x^2 e^{2x} + x^2,$$

the assumed solution is of the form

$$y_p = Ax^5 e^{2x} + Bx^4 e^{2x} + Cx^3 e^{2x} + Dx^2 e^{2x} + Exe^{2x} + Fe^{2x} + Gx^2 + Hx + K,$$

where the first six terms arise from the fact that $e^{2x}$ is a part of the complementary function corresponding to the triple root $m = 2$. Note that $x^{t+s} u = x^{2+3} e^{2x} = x^5 e^{2x}$ and all terms arising from it by differentiation have been used.

**Remarks**

(i) Before constructing the form of the particular integral $y_p$, the solution of the associated homogeneous equation (i.e., roots of the characteristic polynomial) must also be considered.

(ii) We can also use the superposition principle. If $X(x) = f_1(x) + f_2(x) + \ldots + f_r(x)$, we write the form of $y_p$ for each $f_i(x)$ and then add up all these forms o obtain the trial $y_p$.

Based on the ideas discussed so far, it is now possible to state briefly the method of construction for $y_p$ in the form of rules as follows:

**I.** If $X(x)$ is an mth-degree polynomial in x, then $y_p$ is of the form

$x^k (A_o + A_1x + A_2x^2 + \ldots + A_mx^m)$, where k is the multiplicity of 0 as a root of the characteristic polynomial.

**II.** If $X(x)$ is a constant times $e^{\alpha x}$, then $y_p$ is of the form $Ax^k e^{\alpha x}$, where k is the multiplicity of $\alpha$ as a root of the characteristic polynomial.

**III.** If $X(x)$ is $p(x) e^{\alpha x}$, where $p(x)$ is a polynomial of degree m, then $y_p$ has the form

$$x^k (A_o + A_1x + A_2x^2 + \ldots + A_mx^m) e^{\alpha x},$$

where k is the multiplicity of $\alpha$ as a root of the characteristic polynomial. It may be noted that rule III becomes rule I for $\alpha = 0$ and rule II for $m = 0$.

**IV.** If $X(x) = A \cos \beta x + B \sin \beta x$, where at least one of the constants A, B is non-zero, then $y_p$ is of the form

$$x^k (A \cos \beta x + B \sin \beta x),$$

and k is the multiplicity of $\beta i$ as a root of the characteristic polynomial.

**V.** If $X(x) = p(x) \sin \beta x + q(x) \cos \beta x$, where $p(x)$ and $q(x)$ are polynomials in x of degree m and n respectively, then $y_p$ has the form

$$x^k [(A_o + A_1x + A_2x^2 + \ldots + A_sx^s) \cos \beta x + (B_o + B_1x + B_2x^2 + \ldots + B_sx^s) \sin \beta x],$$

where k is the multiplicity of $\beta i$ as a root of the characteristic polynomial and s is the larger of m, n. Note that this rule becomes rule IV for $m = 0$, $n = 0$ and rule I for $\beta = 0$, $n = 0$.

**VI.** If $X(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$, where at least one of the constants A, B is non-zero, then $y_p$ has the form

$$x^k [Ee^{\alpha x} \cos \beta x + Fe^{\alpha x} \sin \beta x].$$

where k is the multiplicity of $\alpha + i\beta$ as a root of the characteristic polynomial.

**Example 1**

Find the general solution of

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 4e^{2x} \qquad \ldots\ldots (i)$$

**Solution**

The associated homogeneous equation corresponding to (i) is

$$y'' - y' - 6y = 0$$

The roots of characteristic polynomial $m^2 - 5x - 6 = (m - 6)(m + 1)$ are $6, -1$.

Therefore, $$y_c = C_1 e^{6x} + C_2 e^{-x}$$

Since derivatives of $e^{2x}$ are always constant multiples of $e^{2x}$ and it is not a term contained in $y_c$, we guess that there is some constant A such that $y_p = Ae^{2x}$ is our trial solution. Substituting in (i), we have

$$(Ae^{2x})'' - 5(Ae^{2x})' - 6(Ae^{2x}) = 4e^{2x}$$

$$4Ae^{2x} - 10Ae^{2x} - 6Ae^{2x} = 4e^{2x}$$

$$-12A = 4 \Rightarrow A = -\tfrac{1}{3} \Rightarrow y_p = -\tfrac{1}{3}e^{2x}$$

Thus the general solution of given equation is

$$y = y_c + y_p = C_1 e^{6x} + C_2 e^{-x} - \tfrac{1}{3}e^{2x}$$

**Example 2**

Solve $$y'' + 2y' + 2y = 3e^{-x} + 4\cos x \qquad \ldots\ldots (ii)$$

**Solution**

The characteristic equation $m^2 + 2m + 2 = 0$ has roots $m = -1 \pm i$. Therefore, $y_c = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$.

For assuming $y_p$ we note that X is sum of two terms, $3e^{-x}$ and $4\cos x$. We first consider $3e^{-x}$. Since $-1$ is not a root of the characteristic equation, $y_p$ will include a term of the form $Ae^{-x}$. Then we consider $4\cos x$. Since i is not a root of the characteristic equation, so $y_p$ will include terms of the form $B\cos x + C\sin x$. Thus the trial form of $y_p$ is

$$y_p = Ae^{-x} + B\cos x + C\sin x,$$

where A, B, C are undetermined coefficients.

Substituting in the original equation (ii), we obtain

$$(Ae^x - B\cos x - C\sin x) + 2(-Ae^x - B\sin x + C\cos x) + Ae^x + 2(Ae^x + B\cos x + C\sin x) = 3e^x + 4\cos x.$$

Equating the coefficients of like terms on both sides of this identity, we get

$$\left.\begin{array}{l} e^x : A = 3 \\ \cos x : B + 2C = 4 \\ \sin x : C - 2B = 0 \end{array}\right\} \Rightarrow A = 3,\ B = \frac{4}{5},\ C = \frac{8}{5}$$

Thus $$y_p = 3e^x + \frac{4}{5}\cos x + \frac{8}{5}\sin x$$

and the solution of (ii) is

$$y = y_c + y_p = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x + 3e^x + \frac{4}{5}\cos x + \frac{8}{5}\sin x$$

**Example 3**

Using the method of undetermined coefficients, solve

$$y'' - 2y' - 3y = 2e^{3x} \qquad \ldots\ldots \text{(iii)}$$

**Solution**

Since the derivative of an exponential such as $e^{3x}$ merely reproduces the function with a possible change in the numerical coefficient, it seems reasonable to assume the form of the desired particular solution as

$$y_p = Ae^{3x},$$

where A is undetermined coefficient to be determined such that $y_p = Ae^{3x}$ is a solution of (iii). Differentiation yields

$$y_p' = 3Ae^{3x} \quad \text{and} \quad y_p'' = 9Ae^{3x}$$

Substituting into (iii), we obtain

$$9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 2e^{3x}$$

or $0.Ae^{3x} = 2e^{3x} \Rightarrow 0 = 2e^{3x}$, which is impossible as it does not hold for any real x. Thus there is no particular solution of the assumed form $y = Ae^{3x}$.

But we can still continue the procedure in this exceptional case and succeed by examining the solutions of the associated homogeneous equation

$$y'' - 2y' - 3y = 0$$

The characteristic equation $m^2 - 2m - 3 = 0$ has roots 3 and $-1$.

Thus $e^{3x}$ and $e^{-x}$ are linearly independent solutions of the homogeneous equation. This suggests that the failure to obtain a solution of the assumed form $y_p = Ae^{3x}$ is due to the fact that the function $e^{3x}$ contained in $y_p$ satisfies the homogeneous equation as well, which a particular solution of (iii) would have to do.

We recall that in the case of a double root m for a characteristic equation, the second solution linearly independent of the basic solution $e^{mx}$ is obtained by multiplying by x. With this as a hint, we consider $y_p = Axe^{3x}$, as a substitute trial solution.

Differentiating, $y_p' = Ae^{3x} + 3Axe^{3x}$ and $y_p'' = 6Ae^{3x} + 9Axe^{3x}$

Substitution into (iii) gives

$$(6Ae^{3x} + 9Axe^{3x}) - 2(Ae^{3x} + 3Axe^{3x}) - 3Axe^{3x} = 2e^{3x}$$

or $$(9 - 6 - 3)\,Axe^{3x} + 4Ae^{3x} = 2e^{3x}$$

which is an identity for all real x. So equating coefficients of $e^{3x}$ on both sides, we get

$$4A = 2 \Rightarrow A = \tfrac{1}{2} \Rightarrow y_p = \tfrac{1}{2}xe^{3x}$$

Also $$y_c = C_1 e^{3x} + C_2 e^{-x}$$

Hence the general solution of (iii) is

$$y = C_1 e^{3x} + C_2 e^{-x} + \tfrac{1}{2}xe^{3x}$$

**Example 4**

Find the trial form for $y_p$ if

$$y'' - y' = (x^3 + x) + (1 - 2x)e^x \qquad \ldots\ldots (iv)$$

**Solution**

The roots of the characteristic polynomial

$$m^2 - m = m(m - 1) \text{ are } m = 0, 1.$$

Thus $$y_c = C_1 e^{0x} + C_2 e^x = C_1 + C_2 e^x$$

Now the first bracket in X(x) of (iv) is a third-degree polynomial. Since derivatives of polynomials are polynomials so we can guess that some polynomial of the form $A + Bx + Cx^2 + Dx^3$ should be contained in $y_p$. Note that, although $x^3 + x$ has a zero constant term, we cannot assume that $y_p$ will also have a zero constant term. So we have included the constant A which may or may not be zero.

Moreover, since 0 is a root of multiplicity 1 of the characteristic equation, therefore, $y_p$ must include a term of the form $x^1(A + Bx + Cx^2 + Dx^3)$. The second

term $(1 - 2x)e^x$ of $X(x)$ is of the form $p(x)\, e^{\alpha x}$, where $p(x) = 1 - 2x$ is a first degree polynomial and $\alpha = 1$ is a root of the characteristic equation, so by rule III, $y_p$ also includes a modified term of the form $x^1(E + Fx)e^x$.

Hence the form of $y_p$ we are searching for is

$$y_p = x(A + Bx + Cx^2 + Dx^3) + x(E + Fx)e^x,$$

where A, B, C etc. are the undetermined coefficients that can be determined in such a way that $y_p$ satisfies (iv).

**Example 5**

Using rules I to VI, find the trial form for $y_p$ if

(i) $y'' + 4y = x^2 \cos 2x + (1 - x) \sin 2x$

(ii) $y'' + 2y' + 2y = 5e^x \cos x$

**Solution**

(i) The roots of the characteristic polynomial $m^2 + 4$ are $\pm 2i$.

Thus $$y_c = C_1 \cos 2x + C_2 \sin 2x$$

Also $X = x^2 \cos 2x + (1 - x) \sin 2x$ is of the form $p(x) \cos \beta x + q(x) \sin \beta x$, where $p(x) = x^2$ is a polynomial of degree two, $g(x) = 1 - x$ is a first degree polynomial and $\beta = 2$. Since $\beta i = 2i$ is a root of the characteristic equation of multiplicity 1, so taking $k = 1$, by rule V the required form of the particular integral is

$$y_p = x[(A + Bx + Cx^2) \cos 2x + (D + Ex + Fx^2) \sin 2x]$$

(ii) The roots of the characteristic polynomial $m^2 + 2m + 2$ are $-1 \pm i$. Thus

$$y_c = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$$

Also X is of the form $e^{\alpha x} \cos \beta x$, where $\alpha = -1$, $\beta = 1$. Since $(-1 + i)$ is a root of the characteristic equation of multiplicity 1, so by rule VI with $k = 1$, the trial form of particular integral is

$$y_p = x(Ae^{-x} \cos x + B e^{-x} \sin x)$$

**Example 6**

Find a particular solution of $y'' + 3y' - y = 7e^{2x} \cos 4x$.

**Solution**

The roots of characteristic equation $m^2 + 3m - 1 = 0$ are $-\frac{3}{2} \pm \frac{\sqrt{13}}{2}$.

We assume the particular integral to be

$$y_p = Ae^{2x} \sin 4x + Be^{2x} \cos 4x$$

Since neither $e^{2x} \sin 4x$ nor $e^{2x} \cos 4x$ is a solution of the associated homogeneous equation $y'' + 3y' - y = 0$, our choice of $y_p$ will work. Subsfituting $y_p$, $y_p'$, $y_p''$ in the given equation, we have

$$(-7A + 28B)e^{2x}\cos 4x + (-28A - 7B)e^{2x}\sin 4x = 7e^{2x}\cos 4x$$

Equating coefficients of like terms on both sides,

$$\left.\begin{aligned} -7A + 28B &= 7 \\ -28A - 7B &= 0 \end{aligned}\right\} \Rightarrow A = -\frac{49}{833} \text{ and } B = \frac{196}{833}$$

Required particular solution is

$$y_p = \frac{1}{833}\left[-49\,e^{2x}\cos 4x + 196\,e^{2x}\sin 4x\right]$$

## EXERCISE 8.6.1

Use the method of undetermined coefficients to solve

1. $y'' + 2y' + 5y = 16e^x + \sin 2x$
2. $y'' - 3y' = 2x^2 + 1$
3. $y'' - 2y' + y = 7xe^x$
4. $y'' - 3y' + 7y = x - \cos 2x$
5. $y'' + 2y' + y = -3e^{-x} + 8xe^{-x} + 1$
6. $y'' - 2y' = e^x \sin x$
7. $y''' + 2y'' - y' - 2y = e^x + x^2$
8. $y'' - 2y' + 3y = x^3 + \sin x$
9. $y'' - 4y' + 4y = x^3e^{2x} + xe^{2x}$
10. $y'' + 4y = x^2 \sin 2x$
11. $y'' + y = 12\cos^2 x$
12. $y'' - 5y' + 6y = 4e^{2x}, \quad y(0) = 0, y'(0) = 1$
13. $y'' - 2y' - 3y = 2e^x - 10\sin x, \quad y(0) = 2, y'(0) = 4$
14. $y'' + 9y = -e^{4x} + x^2\cos 3x$
15. Find the trial form for particular integral of
    (i) $y^{iv} + 3y'' - 4y = \cos^2 x - \cosh x$
    (ii) $y'' + 6y' + 13y = xe^{3x}\sin 2x + x^2e^{2x}\sin 3x$

## 8.7 EXACT LINEAR DIFFERENTIAL EQUATIONS

The differential equation of the type

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x\right) = \phi(x) \qquad \ldots\ldots \text{(i)}$$

is said to be exact if it can be obtained by differentiating once an equation

$$g\left(\frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}, \ldots, \frac{dy}{dx}, y, x\right) = \int \phi(x)\,dx + c \qquad \ldots\ldots \text{(ii)}$$

of next lower order.

Equation (ii) is known as the first integral of equation (i). In case equation (ii) also happens to be exact, then it can be derived simply by differentiation from an equation of one lower order called the second integral of equation (i). In general, there will be n integrals for a differential equation of the nth order.

## 8.7.1 CRITERION FOR EXACTNESS

We shall prove that "the n-th order linear differential equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = \phi(x) \qquad \ldots\ldots \text{(iii)}$$

where the coefficients $P_0, P_1, \ldots, P_n$ are functions of x, is exact provided

$$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots + (-1)^n P_0^{(n)} = 0, \text{ identically}".$$

Let the equation (iii) be exact. Then it can be obtained from an equation of next lower order by differentiating once. But its first term $P_0 \frac{d^n y}{dx^n}$ can be obtained from $P_0 \frac{d^{n-1} y}{dx^{n-1}}$ simply by direct differentiation. So let the first integral of (iii) be

$$P_0 \frac{d^{n-1} y}{dx^{n-1}} + Q_1 \frac{d^{n-2} y}{dx^{n-2}} + Q_2 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + Q_{n-1} y = \int \phi(x)\,dx + c \qquad \ldots\ldots \text{(iv)}$$

where $Q_1, Q_2, \ldots, Q_{n-1}$ are functions of x.

Hence $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y$

$$\equiv \frac{d}{dx}\left[P_0 \frac{d^{n-1} y}{dx^{n-1}} + Q_1 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + Q_{n-1} y\right]$$

$$\equiv \left(P_0 \frac{d^n y}{dx^n} + P'_0 \frac{d^{n-1} y}{dx^{n-1}}\right) + \left(Q_1 \frac{d^{n-1} y}{dx^{n-1}} + Q'_1 \frac{d^{n-2} y}{dx^{n-2}}\right) + \cdots + \left(Q_{n-1} \frac{dy}{dx} + Q'_{n-1} y\right)$$

$$\equiv P_0 \frac{d^n y}{dx^n} + (P'_0 + Q_1) \frac{d^{n-1} y}{dx^{n-1}} + (Q'_1 + Q_2) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + (Q'_{n-2} + Q_{n-1}) \frac{dy}{dx} + Q'_{n-1} y \quad \ldots \text{(v)}$$

where primes denote derivatives w.r.t. x.

Equating coefficients of like terms on both sides of (v), we have

$$P_1 = P_0' + Q_1$$
$$P_2 = Q_1' + Q_2$$
$$P_3 = Q_2' + Q_3$$
$$\cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots$$
$$P_{n-1} = Q_{n-2}' + Q_{n-1}$$
$$P_n = Q_{n-1}'$$

From the above set of equations we can express Q's in terms of P's as

$$\left.\begin{aligned}
Q_1 &= P_1 - P_0 \\
Q_2 &= P_2 - Q_1' = P_2 - (P_1 - P_0')' = P_2 - P_1' + P_0'' \\
Q_3 &= P_3 - Q_2' = P_3 - (P_2 - P_1' + P_0'')' = P_3 - P_2' + P_1'' - P_0''' \\
&\cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots \\
Q_{n-1} &= P_{n-1} - Q_{n-2}' = P_{n-1} - P_{n-2}' + P_{n-3}'' - \cdots + (-1)^{n-1} P_0^{(n-1)}
\end{aligned}\right\} \quad \text{...(vi)}$$

Thus, finally we have

$$P_n = Q_{n-1}' = \left[P_{n-1} - P_{n-2}' + P_{n-3}'' - \cdots + (-1)^{n-1} P_0^{(n-1)}\right]'$$

$$= P_{n-1}' - P_{n-2}'' + P_{n-3}''' - \cdots + (-1)^{n-1} P_0^{(n)}$$

or $$P_n - P_{n-1}' + P_{n-2}'' - P_{n-3}''' + \cdots - (-1)^{n-1} P_0^{(n)} = 0$$

or $$P_n - P_{n-1}' + P_{n-2}'' - P_{n-3}''' + \cdots + (-1)^n P_0^{(n)} = 0 \qquad \cdots\cdots \text{(vii)}$$

which is the required condition of exactness of equation (iii). Hence the first integral of (iii) will be obtained by substituting the value of $Q_1, Q_2, \ldots, Q_{n-1}$ from (vi) in (iv) and is

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P_0') \frac{d^{n-2}y}{dx^{n-2}} + (P_2 + P_1' + P_0'') \frac{d^{n-3}y}{dx^{n-3}} + \cdots$$

$$\cdots + \left[P_{n-1} - P_{n-2}' + P_{n-3}'' - \cdots + (-1)^{n-1} P_0^{(n-1)}\right] y = \int \phi(x)\, dx + c$$

**NOTE** Relation (vii) satisfied by the coefficients in (iii) is both a necessary and sufficient condition for (iii) to be exact.

**EXAMPLE 1** Solve $x(2x+3)\frac{d^2y}{dx^2} + 3(2x+1)\frac{dy}{dx} + 2y = (x+1)e^x$

**SOLUTION** Here $P_0 = 2x^2 + 3x$, $P_1 = 6x + 3$, $P_2 = 2$.

$$\therefore\ P_2 - P_1' + P_0'' = 2 - 6 + 4 = 0$$

Hence the given equation is exact.

Now $P_0 = 2x^2 + 3x$, $Q_1 = P_1 - P_0' = 6x + 3 - (4x + 3) = 2x$

Hence the first integral of the given equation is

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int \phi(x)dx + c$$

i.e., $$(2x^2 + 3x)\frac{dy}{dx} + 2x \cdot y = \int (x+1)e^x dx + c$$

$$= (x+1)e^x - \int e^x dx + c$$

$$= x e^x + e^x - e^x + c$$

$$= x e^x + c$$

or $$\frac{dy}{dx} + \frac{2x}{2x^2 + 3x} \cdot y = \frac{x e^x + c}{2x^2 + 3x}$$

or $$\frac{dy}{dx} + \frac{2}{2x+3} \cdot y = \frac{e^x}{2x+3} + \frac{c}{x(2x+3)}$$

which is a linear equation. Hence its

$$\text{I.F.} = e^{\int \frac{2dx}{2x+3}} = e^{\log(2x+3)} = 2x + 3$$

and the solution is

$$y(2x+3) = \int \frac{(2x+3)e^x}{2x+3}\,dx + \int \frac{c(2x+3)}{x(2x+3)}\,dx + c_1$$

$$= e^x + c\log x + c_1$$

**EXAMPLE 2** Solve $\sin x \frac{d^3y}{dx^3} + (2\cos x + 1) \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = \cos x$ ... (1)

**SOLUTION** Here $P_0 = \sin x,\ P_1 = 2\cos x + 1,\ P_2 = -\sin x,\ P_3 = 0$

$$\therefore \quad P_3 - P_2' + P_1'' - P_0''' = 0 - (-\cos x) + (-2\cos x) - (-\cos x) = 0$$

Hence the given equation is exact.

Also $P_0 = \sin x, \quad Q_1 = P_1 - P_0' = 2\cos x + 1 - \cos x = \cos x + 1$

$$Q_2 = P_2 - P_1' + P_0'' = -\sin x - (-2\sin x) + (-\sin x) = 0$$

Thus the first integral of (1) is

$$P_0 \frac{d^2y}{dx^2} + (P_1 - P_0') \frac{dy}{dx} + 0 = \int \cos x \, dx$$

i.e., $$\sin x \frac{d^2y}{dx^2} + (\cos x + 1) \frac{dy}{dx} = \sin x + c \qquad \ldots\ldots \quad (2)$$

For equation (2), we have

$$P_0 = \sin x,\ P_1 = \cos x + 1,\ P_2 = 0$$

$$\therefore \quad P_2 - P_1' + P_0'' = 0 - (-\sin x) + (-\sin x) = \sin x - \sin x = 0$$

Thus equation (2) is also exact.

Now $P_0 = \sin x,\ Q_1 = P_1 - P_1' = \cos x + 1 - \cos x = 1$

Hence the first integral of (2) i.e., the second integral of (1) is

$$\sin x \frac{dy}{dx} + 1 \cdot y = \int (\sin x + c)\, dx + c_1$$

$$= -\cos x + cx + c_1$$

or $$\frac{dy}{dx} + (\operatorname{cosec} x)\, y = -\cot x + cx \operatorname{cosec} x + c_1 \operatorname{cosec} x$$

which is linear in y. Its

$$\text{I.F.} = e^{\int \operatorname{cosec} x \, dx} = e^{\log \tan \frac{x}{2}} = \tan \frac{x}{2}$$

so that the required solution is

$$y \cdot \tan\frac{x}{2} = \int \tan\frac{x}{2}(cx \operatorname{cosec} x + c_1 \operatorname{cosec} x - \cot x)\,dx + c_2 \qquad \ldots\ldots (3)$$

**NOTE** If the integral cannot be evaluated easily by known standard methods, we give the answer in terms of an integral as in (3) above.

## EXERCISE 8.7

Solve (dashes denote derivatives with respect to x).

**1.** $\sin x\, y'' - \cos x\, y' + 2y \sin x = 0$

**2.** $(x^2 + 1)y'' + 4xy' + 2y = 2\cos x - 2x$

**3.** $x(x^2 - 1)y''' + (8x^2 - 3)y'' + 14x\, y' + 4y = \frac{2}{x^3}$

**4.** $x(x^2 - 4)y''' + 3(3x^2 - 4)y'' + 18x\, y' + 6y = 6$

**5.** $(x + \sin x)y''' + 3(1 + \cos x)y'' - 3\sin x\, y' - y\cos x = -\sin x$

**6.** $x^3y'' + 2xy' - 2y = 1$ $\left(\text{Reducible to exact; I.F.} = \frac{1}{x^2}\right)$

**7.** $y'' = 2y\operatorname{cosec}^2 x$ (Reducible to exact; I.F. = $\cot x$)

**8.** Show that $P_0(x)\, y^{iv} + P_1(x)\, y''' + P_2(x)\, y'' + P_3(x)\, y' + P_4(x)\, y = 0$ is exact if and only if $P_4 - P_3' + P_2'' - P_1''' + P_0^{iv} = 0$

## 8.8 MISCELLANEOUS TYPES OF DIFFERENTIAL EQUATIONS

In this section we propose to study some other types of differential equations of order higher than the first. They require special but useful methods for their solution and turn up often in applications.

### 8.8.1 EQUATIONS IMMEDIATELY INTEGRABLE

i.e., of the form $\frac{d^2y}{dx^2} = f(x)$.

This is an exact differential equation. Since right hand side is a function of x, integrating both sides w.r.t.x, we get

$\frac{dy}{dx} = \int f(x)\,dx + c_1$, which is again exact.

Integrating again w.r.t.x, we have

$$y = \iint f(x)\,(dx)^2 + c_1x + c_2$$

which gives the desired solution.

In general, to solve the equation

$$\frac{d^ny}{dx^n} = f(x)$$

we integrate it n-times successively and get y.

**EXAMPLE** Solve $\frac{d^2y}{dx^2} = x^2 \sin x$

**SOLUTION** Integrating the given equation, we obtain

$$\frac{dy}{dx} = x^2(-\cos x) - \int 2x(-\cos x)\,dx + c_1$$

$$= -x^2 \cos x + 2[x \sin x - \int \sin x\,dx] + c_1$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c_1$$

Integrating again, we have

$$y = -[x^2 \sin x - \int 2x \sin x\,dx] + 2\int x \sin x\,dx + 2\sin x + c_1 x + c_2$$

$$= -x^2 \sin x + 4 \int x \sin x\,dx + 2 \sin x + c_1 x + c_2$$

$$= -x^2 \sin x + 4[x(-\cos x) + \sin x] + 2\sin x + c_1 x + c_2$$

$$= -x^2 \sin x - 4x \cos x + 6 \sin x + c_1 x + c_2$$

which is the required general solution.

## 8.8.2 EQUATIONS OF THE FORM $\frac{d^2y}{dx^2} = f(y)$

This type of equations occur quite frequently in dynamics. In order to solve such an equation, we multiply both sides by $2\frac{dy}{dx}$ and get

$$2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 2f(y) \cdot \frac{dy}{dx}$$

Integrating w.r.t.x, we obtain

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c_1 = \phi(y) + c_1, \text{ say}$$

or $$\frac{dy}{dx} = \sqrt{\phi(y) + c_1}$$

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{\phi(y) + c_1}} = \int dx + c_2 = x + c_2$$

from which the desired solution will follow.

**EXAMPLE** Solve $\frac{d^2y}{dx^2} = \sec^2 y \tan y$ subject to the condition that

$y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

**SOLUTION** Multiplying the given equation by $2\frac{dy}{dx}$, we have

$$2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 2\sec^2 y \tan y \cdot \frac{dy}{dx}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\sec^2 y \tan y \frac{dy}{dx}$$

Integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = 2\int \sec y(\sec y \tan y)dy + c_1 = \sec^2 y + c_1 \quad \ldots\ldots (1)$$

Substituting $y = 0$, $\frac{dy}{dx} = 1$ in (1), we get $c_1 = 0$

Thus $$\left(\frac{dy}{dx}\right)^2 = \sec^2 y \Rightarrow \frac{dy}{dx} = \sec y$$

Separating the variables, we have

$$\cos y\, dy = dx \Rightarrow \sin y = x + c_2 \quad \ldots\ldots (2)$$

Putting $x = 0$, $y = 0$ equation (2) gives $c_2 = 0$.

Hence the required solution is

$$\sin y = x \Rightarrow y = \sin^{-1} x$$

## 8.8.3 EQUATIONS HAVING ONE VARIABLE MISSING

We shall next discuss a method of solving equations in which either the dependent variable y or the independent variable x is missing. The procedure consists in obtaining from the given equation another of lower order by suitable substitutions and then solving the reduced equation by methods already done.

### (a) DEPENDENT VARIABLE ABSENT

Consider the typical second order differential equation

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0 \qquad \ldots\ldots \text{(i)}$$

which does not contain the dependent variable y. To solve it, we put $\frac{dy}{dx} = p$ so that $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. Then the order of (i) is reduced by one and it takes the form

$$f\left(\frac{dp}{dx}, p, x\right) = 0 \qquad \ldots\ldots \text{(ii)}$$

Equation (ii) is of first order in p and x which can be solved for y as a function of x.

If the equation is of the form

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{d^ry}{dx^r}, x\right) = 0$$

i.e., it involves the lowest derivative of order r then we substitute $\frac{d^ry}{dx^r} = p$ to depress the order by one and proceed to solve the reduced equation for p. Then method discussed above in 8.8.1 is applied to find y.

**EXAMPLE** Solve $x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 12x^3$; $y(1) = 0, y'(1) = 1, y''(1) = 0$.

**SOLUTION** Here the dependent variable y is absent from the equation and the lowest order derivative is $\frac{d^2y}{dx^2}$. So we let $\frac{d^2y}{dx^2} = p$. Then, we have from the given equation

$$x\frac{dp}{dx} - 2p = 12x^3 \Rightarrow \frac{dp}{dx} - \frac{2}{x}p = 12x^2$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

$$\therefore \quad p \cdot \frac{1}{x^2} = \int 12x^2 \cdot \frac{1}{x^2}\,dx = 12x + c$$

$$\text{or} \qquad p = 12x^3 + cx^2 \qquad \ldots\ldots (1)$$

But $p = 0$ when $x = 1$. Therefore, $0 = 12 + c \Rightarrow c = -12$. Thus equation (1) becomes

$$p = 12x^3 - 12x^2$$

$$\text{or} \quad \frac{d^2y}{dx^2} = 12x^3 - 12x^2$$

Hence, upon integration we obtain

$$\frac{dy}{dx} = 12\left(\frac{x^4}{4}\right) - 12\left(\frac{x^3}{3}\right) + c_1 = 3x^4 - 4x^3 + c_1 \qquad \ldots\ldots (2)$$

Applying $\frac{dy}{dx} = 1$ when $x = 1$, we have

$$1 = 3 - 4 + c_1 \Rightarrow c_1 = 2$$

$\therefore$ Equation (2) becomes

$$\frac{dy}{dx} = 3x^4 - 4x^3 + 2$$

$$\Rightarrow \quad y = 3\left(\frac{x^5}{5}\right) - 4\left(\frac{x^4}{4}\right) + 2x + c_2 \qquad \ldots\ldots (3)$$

But $y = 0$ when $x = 1$. Hence $c_2 = -\frac{8}{5}$.

$\therefore$ From (3), $y = \frac{3}{5}x^5 - x^4 + 2x - \frac{8}{5} = \frac{1}{5}(3x^5 - 5x^4 + 10x - 8)$ which gives the required solution.

### (b) INDEPENDENT VARIABLE ABSENT

If the independent variable $x$ is missing, then the equation has the form

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y\right) = 0$$

We can transform it into an equation of reduced order by substituting $\frac{dy}{dx} = p$.

However, to get an equation in y and p we have to express $\frac{d^r y}{dx^r}$ in terms of p and its derivatives w.r.t.y. For achieving this purpose we may write

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\cdot\frac{dp}{dy},$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\,\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\,\frac{dp}{dy}\right)\cdot\frac{dy}{dx} = \left[p\,\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right]\cdot\frac{dy}{dx}$$

$$= p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \text{ etc.}$$

These substitutions will reduce the order of the given differential equation by one and the reduced equation will involve two variables p and y and can be solved further by the known methods.

**NOTE** If an equation is free of x and y both, we can apply either (*a*) or (b).

**EXAMPLE** Solve $(1 + y^2)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} = 0$

**SOLUTION** In this case the independent variable x is missing.

Let $\frac{dy}{dx} = p$. Then $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$ and the given equation reduces to

$$(1 + y^2)\,p\frac{dp}{dy} + p^3 + p = 0$$

or $$(1 + y^2)\frac{dp}{dy} + (p^2 + 1) = 0$$

or $$\frac{dp}{p^2 + 1} + \frac{dy}{y^2 + 1} = 0$$

Integration yields $\tan^{-1}p + \tan^{-1}y = c$

or $$\tan^{-1}\left(\frac{p + y}{1 - py}\right) = c$$

or $$\frac{p + y}{1 - py} = \tan c = c_1, \text{ say}$$

or $\quad p + y = c_1 - c_1 py$

or $\quad \dfrac{dy}{dx} = p = \dfrac{c_1 - y}{1 + c_1 y}$

Separating the variables, we obtain

$$\frac{c_1 y + 1}{y - c_1} dy = -dx$$

or $$\int \left( c_1 + \frac{1 + c_1^2}{y - c_1} \right) dy = -\int dx$$

$$\Rightarrow \quad c_1 y + (1 + c_1^2) \log (y - c_1) = -x + c_2$$

or $\quad x + c_1 y + (1 + c_1^2) \log (y - c_1) = c_2$

which gives the required solution.

## EXERCISE 8.8

Solve

**1.** $\dfrac{d^3y}{dx^3} = \log x$ **2.(i)** $y^3 \dfrac{d^2y}{dx^2} + a^2 = 0$; (ii) $\dfrac{d^2y}{dx^2} + m^2 y = 0$

**3.** $\dfrac{d^2y}{dx^2} = 2y(y^2 + 1)$, given that $y = 0$, $\dfrac{dy}{dx} = 1$ when $x = 0$

**4.** $\dfrac{dy}{dx} = x \dfrac{d^2y}{dx^2} - 3x^2$

**5.** $\dfrac{d^2y}{dx^2} = \left( \dfrac{dy}{dx} \right)^2 + 1$

**6.** $2y \dfrac{d^2y}{dx^2} = 1 + \left( \dfrac{dy}{dx} \right)^2$

**7.** $y \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} = 2 \left( \dfrac{dy}{dx} \right)^2$

**8.** $x \dfrac{d^3y}{dx^3} = 2 \dfrac{d^2y}{dx^2}$

**9.** $y \dfrac{d^2y}{dx^2} + \left( \dfrac{dy}{dx} \right)^2 = \dfrac{dy}{dx}$

**10.** $x^2 \left( \dfrac{d^2y}{dx^2} \right)^2 = 1 + \left( \dfrac{dy}{dx} \right)^2$

**11.** $(1 - x^2) \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} = 2$

**12.** $(1 + x^2) \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + 4x = 0$

**13.** $\left( \dfrac{d^3y}{dx^3} \right)^2 + x \dfrac{d^3y}{dx^3} = \dfrac{d^2y}{dx^2}$

**14.** $y \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \left( \dfrac{dy}{dx} - 2y \right) = 0$

**15.** $y \dfrac{d^2y}{dx^2} - \left( \dfrac{dy}{dx} \right)^2 = y^2 \log y$

**16.** $x \dfrac{d^2y}{dx^2} - \left( \dfrac{dy}{dx} \right)^3 - \dfrac{dy}{dx} = 0$

# 9

# DIFFERENTIAL EQUATIONS – III

## 9.1 SIMULTANEOUS DIFFERENTIAL EQUATIONS

So far we have studied differential equations which involved one independent and one dependent variable. But differential equations containing one independent and two or more dependent variables often arise in applications. For example, in

$$\frac{dx}{dt} + \frac{dy}{dt} + y = 2e^{2t} \quad \ldots\ldots \text{(i)}$$

two variables x and y depend upon a single independent variable t.

If we can find x and y in terms of t, then the pair of functions $x = f(t)$ and $y = g(t)$ which when substituted in (i) reduces it to an identity, is called its solution. We can find an infinite number of such pairs. For example, one such pair is obtained by substituting $y = 3t^2$ in (i), yielding

$$\frac{dx}{dt} = 2e^{2t} - 6t - 3t^2$$

$$\text{i.e.,} \quad x = e^{2t} - 3t^2 - t^3 + c$$

We can limit the number of possible pairs by considering another differential equation, say

$$\frac{dy}{dt} = x + y \quad \ldots\ldots \text{(ii)}$$

and then determining pairs $x = f(t)$ and $y = g(t)$ which satisfy simultaneously equations (i) and (ii). For this reason the equations (i) and (ii) are called simultaneous differential equations.

## 9.1.1 DETERMINATION OF SOLUTION

We shall now focus our attention to the solution of a system of simultaneous linear differential equations with constant coefficients. The complete solution can only be obtained if the number of equations in the system is equal to the number of dependent variables. The method of solving such differential equations is analogous to the one adopted in algebra for solving simultaneous equations by systematic elimination of variables. That is, by using process of elimination we obtain an equation involving one dependent variable and one independent variable. After solving this derived equation by methods discussed earlier, we substitute back to find the other dependent variable or variables involved in the given system. In the process of elimination of variables, we shall notice that the differential operators can be used to advantage.

Let us consider the case of simultaneous differential equations involving two dependent variables x and y and one independent variable t. In operator notation these equations can be written as

$$f_1(D)x + g_1(D)y = f(t) \quad \ldots\ldots \text{(iii)}$$

$$f_2(D)x + g_2(D)y = g(t) \quad \ldots\ldots \text{(iv)}$$

where $D \equiv \frac{d}{dt}$ and $f_1(D)$, $g_1(D)$, $f_2(D)$ and $g_2(D)$ are polynomial operators with constant coefficients.

To eliminate one of the dependent variables, say y, we operate on both sides of (iii) by $g_2(D)$ and (iv) by $g_1(D)$ and subtract to obtain

$$[f_1(D)g_2(D) - g_1(D)\, f_2(D)]x = g_2(D)\, f(t) - g_1(D)\, g(t)$$

or

$$\phi(D)x = \psi(t)$$

This is an ordinary differential equation with one dependent variable x and can be solved by methods already considered, yielding x as a function of independent variable t. By substituting the value of x so obtained in the given equations (iii) or (iv) we can get the value of the other dependent variable y in terms of t.

If y is obtained by an independent elimination as we did in the case of x, then the values of x and y are substituted in the given equations (iii) and (iv) and the arbitrary constants involved in the simultaneous solution $x = \phi(t)$, $y = \psi(t)$ are to be so adjusted that the given equations are satisfied. It can be proved that the number of independent arbitrary constants appearing in the general solution of the given system of linear equations with constant coefficients is equal to the index of the highest power of D in the polynomial operator obtained from the determinant of the coefficients of x and y

$$\text{i.e.,} \quad \Delta(D) = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix},$$

provided that this determinant is not zero.

**EXAMPLE 1** Solve simultaneously

$$\frac{dx}{dt} + 3\frac{dy}{dt} + y = e^t \qquad \ldots\ldots (1)$$

$$\frac{dy}{dt} - x = y \qquad \ldots\ldots (2)$$

**SOLUTION** To eliminate one of the dependent variables, from equation (2) we have

$$x = \frac{dy}{dt} - y \qquad \ldots\ldots (3)$$

Substituting in (1) we get

$$\frac{d}{dt}\left(\frac{dy}{dt} - y\right) + 3\frac{dy}{dt} + y = e^t$$

or $$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 3\frac{dy}{dt} + y = e^t$$

or $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^t \qquad \ldots\ldots (4)$$

which involves only one dependent variable y. Its auxiliary equation is

$$D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1.$$

$$\therefore \quad y_c = (c_1 + c_2 t)e^{-t}$$

Also $$y_p = \frac{1}{D^2 + 2D + 1} e^t = \frac{1}{4} e^t$$

Thus the complete solution of (4) is

$$y = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{4} e^t \qquad \ldots\ldots (5)$$

Hence from (3) we have

$$x = \frac{dy}{dt} - y$$

$$= \frac{d}{dt}\left(c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^t\right) - c_1e^{-t} - c_2te^{-t} - \frac{1}{4}e^t$$

$$= -c_1e^{-t} + c_2(-te^{-t} + e^{-t}) + \frac{1}{4}e^t - c_1e^{-t} - c_2te^{-t} - \frac{1}{4}e^t$$

$$= -2c_1e^{-t} + c_2e^{-t} - 2c_2te^{-t} \quad \ldots\ldots (6)$$

which gives the second dependent variable x as a function of t.

The pair of equations (5) and (6) constitute the simultaneous solution of the given system.

Alternatively, instead of using (3) if we substitute the value of y from (5) in (1), we get

$$\frac{dx}{dt} = e^t - 3\frac{d}{dt}\left(c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^t\right) - \left(c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^t\right)$$

$$= e^t - 3\left[-c_1e^{-t} - c_2te^{-t} + c_2e^{-t} + \frac{1}{4}e^t\right] - c_1e^{-t} - c_2te^{-t} - \frac{1}{4}e^t.$$

$$= 2c_1e^{-t} - 3c_2e^{-t} + 2c_2te^{-t}.$$

Integrating, we obtain

$$x = -2c_1e^{-t} + 3c_2e^{-t} + 2c_2(-te^{-t} - e^{-t}) + c_3$$

$$= -2c_1e^{-t} + c_2e^{-t} - 2c_2te^{-t} + c_3 \quad \ldots\ldots (7)$$

We must now check to see whether the number of independent arbitrary constants present in the simultaneous solution constituted by (5) and (7) is the same as the degree of the polynomial operator obtained from the coefficient determinant. But the degree of

$$\begin{vmatrix} D & 3D + 1 \\ -1 & D - 1 \end{vmatrix} = D^2 + 2D + 1$$

is 2. So there should be only two constants whereas the number of constants in (7) is 3. That is, one constant is extraneous. Hence we proceed to substitute x and y from (5) and (7) respectively into either of the original equations, say (2) and get

$$\left[-c_1e^{-t} + c_2(-te^{-t} + e^{-t}) + \frac{1}{4}e^{t}\right] - (-2c_1e^{-t} + c_2e^{-t} - 2c_2te^{-t} + c_3)$$

$$= \quad c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^{t}$$

which implies that $c_3 = 0$ so that (7) reduces to (6).

**NOTE** To establish any relation that may exist between the extraneous constants of integration and expected constants we substitute the simultaneous solution in one of the equations of the given system, simplify as much as possible, and then equate the coefficients of like terms in the left and right hand sides of the result.

**EXAMPLE 2** Solve the simultaneous equations

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

subject to the conditions $x = 0,\ y = 0$ when $t = 0$.

**SOLUTION** Let $\frac{d}{dt} = D$. Then the given equations can be written in symbolic form as

$$(D + 5)x - 2y = t \qquad \ldots\ldots (1)$$

$$2x + (D + 1)y = 0 \qquad \ldots\ldots (2)$$

In order to eliminate x we multiply equation (1) by 2 and operate on equation (2) by (D + 5) and obtain

$$2(D + 5)x - 4y = 2t \qquad \ldots\ldots (3)$$

$$2(D + 5)x + (D + 5)(D + 1)y = 0 \qquad \ldots\ldots (4)$$

Subtracting (4) from (3), we get

$$[-4 - (D^2 + 6D + 5)]y = 2t$$

$$(D^2 + 6D + 9)y = -2t$$

Its characteristic equation. $D^2 + 6D + 9 = 0$ gives $D = -3, -3$.

$\therefore \quad y_c = (c_1 + c_2t)e^{-3t}$

Also $y_p = \frac{1}{(D+3)^2}(-2t) = -\frac{2}{9}\left[1+\frac{D}{3}\right]^{-2} t.$

$= -\frac{2}{9}\left[1 - \frac{2D}{3} + \ldots\right] t,$ ignoring $D^2, D^3$ etc.

$= -\frac{2t}{9} + \frac{4}{27}$

$\therefore \quad y = y_c + y_p = (c_1 + c_2 t)e^{-3t} - \frac{2t}{9} + \frac{4}{27}$ ..... (5)

Now to find x we can, as above, eliminate y from equation (1) and (2) and solve the resulting equation. But it is more convenient here to substitute the value of y from (5) in (2) and get

$$2x + (D+1)\left[(c_1 + c_2 t)e^{-3t} - \frac{2t}{9} + \frac{4}{27}\right] = 0$$

$$2x + \left[(c_1 + c_2 t)(-3)e^{-3t} + c_2 e^{-3t} - \frac{2}{9}\right] + \left[(c_1 + c_2 t)e^{-3t} - \frac{2t}{9} + \frac{4}{27}\right] = 0$$

$$2x - 2(c_1 + c_2 t)e^{-3t} + c_2 e^{-3t} - \frac{2t}{9} - \frac{2}{27} = 0$$

$$x = \left[c_1 - \frac{c_2}{2} + c_2 t\right]e^{-3t} + \frac{t}{9} + \frac{1}{27} \quad \ldots\ldots \quad (6)$$

Hence the pair of equations (5) and (6) constitute the simultaneous solution of the given system of equations.

Using $x = 0$ when $t = 0$, equation (6) $\Rightarrow c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0$

Using $y = 0$ when $t = 0$, equation (5) $\Rightarrow c_1 + \frac{4}{27} = 0$

These two equations yield $c_1 = -\frac{4}{27}$ and $c_2 = -\frac{2}{9}$

Thus the required solutions are

$$x = \left(-\frac{1}{27} - \frac{2}{9}t\right)e^{-3t} + \frac{t}{9} + \frac{1}{27} = -\frac{1}{27}[1 + 6t]e^{-3t} + \frac{1}{27}[1 + 3t]$$

and $$y = \left(-\frac{4}{27} - \frac{2}{9}t\right)e^{-3t} - \frac{2t}{9} + \frac{4}{27} = -\frac{2}{27}[2 + 3t]e^{-3t} + \frac{2}{27}[2 - 3t]$$

**EXAMPLE 3** Solve the system

$$(D + 1)x + (D - 1)y = e^t \quad \ldots\ldots (1)$$

$$(D^2 + D + 1)x + (D^2 - D + 1)y = t^2, \quad D = \frac{d}{dt} \quad \ldots\ldots (2)$$

**SOLUTION** For the given equations, the determinant of the operator coefficients of x and y is

$$\Delta = \begin{vmatrix} D + 1 & D - 1 \\ D^2 + D + 1 & D^2 - D + 1 \end{vmatrix}$$

$$= (D + 1)(D^2 - D + 1) - (D - 1)(D^2 + D + 1)$$

$$= D^3 + 1 - D^3 + 1$$

$$= 2$$

Note that the index of the highest power of D in $\Delta$ is zero. Hence there cannot be any independent arbitrary constant present in the general solution of the given simultaneous equations.

To eliminate y we operate on (1) by $(D^2 - D + 1)$ and on (2) by $(D - 1)$ and subtract to have

$$[(D^2 - D + 1)(D + 1) - (D - 1)(D^2 + D + 1)]x = (D^2 - D + 1)e^t - (D - 1)t^2$$

$$[(D^3 + 1) - (D^3 - 1)]x = e^t - e^t + e^t - 2t + t^2$$

$$2x = e^t - 2t + t^2$$

$$x = \frac{1}{2}[e^t - 2t + t^2] \quad \ldots\ldots (3)$$

Now to eliminate x we operate on (1) by $(D^2 + D + 1)$ and on (2) by $(D + 1)$ and subtract:

$$[(D^2 + D + 1)(D - 1) - (D + 1)(D^2 - D + 1)]y = (D^2 + D + 1)e^t - (D + 1)t^2$$

$$[(D^3 - 1) - (D^3 + 1)]y = e^t + e^t + e^t - 2t - t^2$$

$$-2y = 3e^t - 2t - t^2$$

$$y = \frac{1}{2}[t^2 + 2t - 3e^t] \quad \ldots\ldots (4)$$

The pair of equations (3) and (4) constitute the required simultaneous solution.

## EXERCISE 9.1

Solve simultaneously

**1.** $\frac{dx}{dt} + \frac{dy}{dt} + 2y = 0$

$\frac{dx}{dt} - 3x - 2y = 0$

**2.** $\frac{dx}{dt} - 7x + y = 0$

$\frac{dy}{dt} - 2x - 5y = 0$

**3.** $\frac{dx}{dt} + \frac{dy}{dt} - x = 2t + 1$

$2\frac{dx}{dt} + 2\frac{dy}{dt} + x = t$

**4.** $\frac{dx}{dt} + y = \sin t$

$\frac{dy}{dt} + x = \cos t$

given that $x = 2, y = 0$ when $t = 0$.

**5.** $(D + 4)x + 3y = t$

$2x + (D + 5)y = e^t$, $D = \frac{d}{dt}$

**6.** $(D + 1)y = z + e^x$

$(D + 1)z = y + e^x$, $D = \frac{d}{dx}$

**7.** $Dx - 3y = 0$

$Dy + z = 0$

$Dz + y = 0, D = \frac{d}{dt}$

**8.** $(D - 3)x + 2(D + 2)y = 2 \sin t$

$2(D + 1)x + (D - 1)y = \cos t, D = \frac{d}{dt}$

**9.** $(D + 2)x - 3y = t$

$-3x + (D + 2)y = e^{2t}, D = \frac{d}{dt}$

**10.** $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

**11.** $\frac{dx}{dt} - 3x - 6y = t^2$

$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$

**12.** $\frac{dx}{dt} + \frac{dy}{dt} + y - x = e^{2t}$

$\frac{d^2x}{dt^2} + \frac{dy}{dt} = 3e^{2t}$

**13.** $\frac{d^2x}{dt^2} - 3x - 4y = 0$

$\frac{d^2y}{dt^2} + x + y = 0$

**14.** $(D^2 - 4)x - (D + 2)y = 4t$

$(D + 7)x + (D + 3)y = 0$

$x = y = 1$, $Dx = Dy = 0$ when $t = 0$.

**15.** $(D^2 - 2)x - 3y = e^{2t}$

$(D^2 + 2)y + x = 0$, given that

## 9.2 TOTAL DIFFERENTIAL EQUATIONS

We shall now study the ordinary differential equations of the first order and first degree involving a single independent variable and two or more than two dependent variables. Our discussion will be confined to three variables only, although it can be extended to the solution of ordinary differential equations containing more than three variables. An ordinary differential equation of the first order and first degree involving three variables defined by

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0$$

where P, Q and R are differentiable functions of x, y, z and x has been taken as the independent variable, is called a total differential equation. In terms of differentials this equation can equivalently be written as

$$P(x, y, z)\,dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \quad \ldots\ldots \text{(i)}$$

### 9.2.1 CONDITION OF INTEGRABILITY

We shall now determine the condition which P, Q, R should satisfy so that equation (i) may be integrable. Let the differential equation (i) be integrable and

$$u(x, y, z) = c \quad \ldots\ldots \text{(ii)}$$

be its integral, where c is a constant. Then the total differential du must be equal to Pdx + Qdy + Rdz or to it multiplied by an integrating factor. But

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \ldots\ldots \text{(iii)}$$

Comparing the coefficients of dx, dy and dz in (i) and (iii), we obtain

$$\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \mu, \text{ say}$$

where $\mu$ is a function of x, y and z.

$$\therefore \quad \frac{\partial u}{\partial x} = \mu P \quad \ldots\ldots \text{(iv)}$$

$$\frac{\partial u}{\partial y} = \mu Q \quad \ldots\ldots \text{(v)}$$

$$\frac{\partial u}{\partial z} = \mu R \quad \ldots\ldots \text{(vi)}$$

Differentiating equation (iv) w.r.t. y and (v) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y}$$

and $$\frac{\partial^2 u}{\partial x \partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$$

Now assuming the existence and continuity conditions, we know that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

which gives

$$P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$$

or $$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \quad \ldots\ldots \text{(vii)}$$

Similarly, from equations (v) and (vi), (vi) and (iv), we get

$$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \quad \ldots\ldots \text{(viii)}$$

and $$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \quad \ldots\ldots \text{(ix)}$$

Multiplying equations (vii), (viii) and (ix) by R, P and Q respectively and adding, we obtain

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \text{ , since } \mu \neq 0 \quad \ldots\ldots \text{(x)}$$

which is the required condition for the total differential equation (i) to be integrable.

NOTE (i) The relation (x) is both a necessary and sufficient condition for the integrability of the differential equation (i).

(ii) An easy way to remember condition (x) is to notice that it is equivalent to

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$$

### 9.2.2 CONDITIONS FOR EXACTNESS OF (i).

If $\mu(x, y, z) = 1$, then equation (i) is exact and from (vii), (viii) and (ix) we obtain

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0, \quad \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0$$

or $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad \ldots\ldots \text{(xi)}$$

It may be noted that when conditions (xi) are satisfied the condition (x) for integrability of (i) is also satisfied because each term of (x) vanishes identically.

### 9.2.3 METHODS OF SOLVING $Pdx + Qdy + Rdz = 0$

**I. GENERAL METHOD** By assuming one of the variables as constant.

When the condition of integrability of

$$Pdx + Qdy + Rdz = 0 \qquad \ldots\ldots (1)$$

is satisfied we are sure that its primitive exists. We proceed to consider one of the variables say z as constant so that $dz = 0$. A proper choice of this variable is necessary for conveniently solving the resulting equation

$$Pdx + Qdy = 0 \qquad \ldots\ldots (2)$$

Let the integral of (2) be

$$u(x, y) = \text{an arbitrary constant} = \phi(z), \text{ say} \qquad \ldots\ldots (3)$$

Then we take the total differential of (3) and compare the result with the given equation (1). As a result we get a differential equation involving two variables z and $\phi(z)$. In case the coefficient of $d\phi$ or dz involves functions of x and y, it can always be removed by making use of (3). On solving this equation we determine $\phi(z)$ and substitute for $\phi(z)$ in (3) to obtain the general solution of the given total differential equation (1).

**EXAMPLE** Solve $(x + z)^2 dy + y^2 dx + y^2 dz = 0$

**SOLUTION** We can re-write the given equation as

$$y^2 dx + (x + z)^2 dy + y^2 dz = 0$$

Then $P = y^2$, $Q = (x + y)^2$, $R = y^2$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \quad y^2(2x + 2z - 2y) + (x + z)^2 \cdot (0 - 0) + y^2(2y - 2x - 2z)$$

$$= \quad 2xy^2 + 2y^2z - 2y^3 + 2y^3 - 2xy^2 - 2y^2z$$

$$= \quad 0$$

Hence the given equation is integrable.

Let us take y as constant so that $dy = 0$.

Then the given equation reduces to

$$y^2(dx + dz) = 0$$

or $$dx + dz = 0$$

Its solution is $$x + z = \phi(y), \text{ say} \qquad \ldots\ldots (1)$$

Taking differential, we obtain

$$dx + dz = \frac{d\phi}{dy} dy$$

Comparing it with the given equation

$$dx + dz = -\frac{(x + z)^2}{y^2} dy,$$

we have $$\frac{d\phi}{dy} = -\frac{(x + z)^2}{y^2} = -\frac{\phi^2}{y^2}$$ , by using (1).

or $$\frac{d\phi}{\phi^2} + \frac{dy}{y^2} = 0$$

Its solution is $$\frac{-1}{\phi} + \frac{-1}{y} = \text{constant.}$$

or $$\frac{1}{\phi} + \frac{1}{y} = c$$

Substituting the value of $\phi$ from (1) we get

$$\frac{1}{x + z} + \frac{1}{y} = c$$

or $$x + y + z = cy(x + z)$$

which gives the required solution of the given equation.

### 9.2.4 II. METHOD OF REGROUPING THE TERMS (ALSO CALLED THE SOLUTION BY INSPECTION)

When the given equation is exact, an inspection of the equation can help us to rearrange the terms in such a manner that each of the new groups is an exact differential and then the integral can conveniently be obtained. In such cases there is no need to apply the condition of integrability.

Sometimes the given equation may not be exact but is integrable. In that case we regroup the terms into exact differentials after multiplying by a suitable function of x, y, z as an integrating factor.

**EXAMPLE 1** Solve $(\cos x + e^x y)\,dx + (e^x + e^y z)\,dy + e^y dz = 0$

**SOLUTION** Here $P = \cos x + e^x y$, $Q = e^x + e^y z$, $R = e^y$

Since $\frac{\partial P}{\partial y} = e^x = \frac{\partial Q}{\partial x}$; $\frac{\partial Q}{\partial z} = e^y = \frac{\partial R}{\partial y}$; $\frac{\partial R}{\partial x} = 0 = \frac{\partial P}{\partial z}$

Hence the given equation is exact. Regrouping the terms, we have

$$\cos x\,dx + (e^x y\,dx + e^x\,dy) + (e^y z\,dy + e^y\,dz) = 0$$

As each group is an exact differential, so the required integral is

$$\sin x + ye^x + ze^y = c$$

**EXAMPLE 2** Solve $(b - z)(ydx + xdy) + xy\,dz = 0$

**SOLUTION** Here $P = (b - z)y$, $Q = (b - z)x$, $R = xy$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (by + zy)[-x - x] + (bx - zx)[y - (-y)] + xy[b - z - (b - z)]$$

$$= -2bxy + 2xyz + 2bxy - 2xyz$$

$$= 0$$

Thus the given equation is integrable but not exact as $\frac{\partial Q}{\partial z} \neq \frac{\partial R}{\partial y}$. So we divide throughout by $(b - z)\,xy$ as I.F. and get

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{b - z} = 0$$

or $d[\log x + \log y - \log (b - z)] = 0$

Integrating we have

$$\log x + \log y - \log (b - z) = \log c.$$

$$xy = c(b - z)$$

**EXAMPLE 3** Solve $(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0$

**SOLUTION** Here $P = y^2 + z^2 - x^2$, $Q = -2xy$, $R = -2xz$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + z^2 - x^2)\,[0 - 0] + (-2xy)\,[-2z - 2z] + (-2xy)\,[2y - (-2y)]$$

$$= 8x\,yz - 8x\,yz$$

$$= 0.$$

Hence the condition of integrability is satisfied but the equation is not exact. So we make use of a suitable integrating factor and regroup the terms into exact differentials. The given equation can be rearranged as

$$y^2dx - 2xy\,dy + z^2dx - 2xz\,dz - x^2dx = 0$$

$$\frac{y^2dx - 2xy\,dy}{x^2} + \frac{z^2dx - 2xz\,dz}{x^2} - dx = 0$$

$$-\left[\frac{2xy\,dy - y^2dx}{x^2}\right] - \left[\frac{2xz\,dz - z^2dx}{x^2}\right] - dx = 0$$

$$-d\left(\frac{y^2}{x}\right) - d\left(\frac{z^2}{x}\right) - dx = 0$$

Integrating, we get

$$\frac{y^2}{x} + \frac{z^2}{x} + x = c$$

$$x^2 + y^2 + z^2 = cx$$

which is the required solution.

**NOTE** The students are advised to solve it by Method I as well.

## 9.2.5 III. METHOD FOR HOMOGENEOUS EQUATION

Let $P\,dx + Q\,dy + R\,dz = 0$ ..... (1)

be an integrable equation in which the coefficients P, Q, R are homogeneous functions of the the same degree n in x, y and z. Let us make the substitutions.

$$x = uz \quad \text{and} \quad y = vz$$

so that $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$.

Also P, Q, R are homogeneous functions of the same degree n,

$$\therefore \quad P = z^n f\left(\frac{x}{z}, \frac{y}{z}\right) = z^n f(u, v)$$

$$Q = z^n g\left(\frac{x}{z}, \frac{y}{z}\right) = z^n g(u, v)$$

$$R = z^n h\left(\frac{x}{z}, \frac{y}{z}\right) = z^n h(u, v)$$

By the above substitutions equation (1) becomes

$$z^n f[u\,dz + z\,du] + z^n g[v\,dz + z\,dv] + z^n h\,dz = 0$$

$$z^{n+1}[f\,du + g\,dv] + z^n[uf + vg + h]\,dz = 0 \quad \text{..... (2)}$$

Now there are two cases to be considered.

**CASE** (i) If the coefficient of dz in equation (2) is zero, we shall get an equation

$$f\,du + g\,dv = 0 \quad \text{..... (3)}$$

in only two variables u and v and hence can be solved. The solution of (1) will be obtained by substituting x/z for u and y/z for v in the solution of (3).

**CASE** (ii) If the coefficient of dz in (2) is not zero, then we can separate the variable z from the variables u and v by dividing (2) by $z^{n+1}[uf + vg + h]$ and obtain

$$\frac{f\,du + g\,dv}{uf + vg + h} + \frac{dz}{z} = 0 \quad \text{..... (4)}$$

Since the second term in equation (4) is a perfect differential and the condition of integrability has been supposed to be satisfied, therefore, the first term must also be a perfect differential. Hence equation (4) can be integrated. The required solution in x, y, z will be obtained by replacing u and v by x/z and y/z respectively in the solution of equation (4).

## NOTE : INTEGRATING FACTOR FOR HOMOGENEOUS EQUATION

We have seen above that equation (2) becomes exact on dividing by

$$z^{n+1}[uf + vg + h] = z^{n+1}\left[\frac{x}{z}f + \frac{y}{z}g + h\right]$$

$$= x(z^n f) + y(z^n g) + z(z^n h)$$

$$= xP + yQ + zR$$

Hence $\frac{1}{xP + yQ + zR}$ is an integrating factor of the integrable homogeneous equation (1) if $xP + yQ + zR \neq 0$. Note that if $xP + yQ + zR = 0$, then the substitutions $x = uz$ and $y = vz$ will separate the variable z from the variables u and v.

**EXAMPLE 1** Solve $y(y + z)\,dx + z(x + z)dy + y(y - x)dz = 0$

Here $P = y^2 + yz$ , $Q = xz + z^2$ , $R = y^2 - xy$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)[(x + 2z) - (2y - x)] + (xz + z^2)[-y - y] + (y^2 - xy)[(2y + z) - z]$$

$$= 2(y^2 + yz)(x + z - y) - 2y(xz + z^2) + 2y(y^2 - xy)$$

$$= 2xy^2 + 2y^2z - 2y^3 + 2xyz + 2yz^2 - 2y^2z - 2xyz - 2yz^2 + 2y^3 - 2xy^2$$

$$= 0$$

Hence the given equation is integrable. Also it is homogeneous of degree 2. So on putting

$x = uz$ so that $dx = udz + zdu$

and $y = vz$ so that $dy = vdz + zdv$

the given equation is transformed to

$$z^2(v^2 + v)[udz + zdu] + z^2(u + 1)[vdz + zdv] + z^2(v^2 - uv)\,dz = 0$$

or $$z^3(v^2 + v)\,du + z^3(u + 1)\,dv + z^2[u(v^2 + v) + v(u + 1) + (v^2 - uv)]\,dz = 0$$

or $$z^3(v^2 + v)du + z^3(u + 1)dv + z^2[uv^2 + uv + uv + v + v^2 - uv]\,dz = 0$$

or $z^3(v^2 + v)du + z^3(u + 1)dv + z^2(u + 1)(v^2 + v)\,dz = 0$

Here the coefficient of dz is not zero. Dividing throughout by $z^3(u+1)(v^2+v)$ we separate z from u and v and obtain

$$\frac{du}{u + 1} + \frac{dv}{v(u + 1)} + \frac{dz}{z} = 0$$

or $$\frac{du}{u + 1} + \left(\frac{1}{v} - \frac{1}{v + 1}\right)dv + \frac{dz}{z} = 0$$

Integrating we obtain

$$\log (u + 1) + \log v - \log (v + 1) + \log z = \log C$$

$$vz (u + 1) = C (v + 1)$$

Replacing u by $\frac{x}{z}$ and v by $\frac{y}{z}$, we get

$$\frac{y}{z} \cdot z \cdot \left(\frac{x}{z} + 1\right) = C\left(\frac{y}{z} + 1\right)$$

$$y(x + z) = C(y + z)$$

which is the required solution.

**NOTE** Since $xP + yQ + zR = y(x + z)(y + z) \neq 0$, so by taking $\frac{1}{xP + yQ + zR}$ as integrating factor, we can solve the above equation without using the transformations $x = uz$ and $y = vz$.

**EXAMPLE 2** Solve $yz^2(x^2 - yz)\,dx + zx^2(y^2 - zx)dy + xy^2(z^2 - xy)dz = 0$

**SOLUTION** Here $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (yz^2x^2 - y^2z^3)[x^2y^2 - 2zx^3 - 2xyz^2 + 3x^2y^2]$$

$$+(zx^2y^2 - z^2x^3)[y^2z^2 - 2xy^3 - 2yzx^2 + 3y^2z^2]$$

$$+(xy^2z^2 - x^2y^3)[z^2x^2 - 2yz^3 - 2zxy^2 + 3z^2x^2]$$

$$= 0$$

Hence the given equation is integrable. Also it is homogeneous, so we make the substitution

$$x = uz \text{ and } y = vz$$

so that $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$. Then the given equation becomes

$$(z^5vu^2 - z^5v^2)[u\,dz + z\,du] + (z^5u^2v^2 - z^5u^3)[v\,dz + z\,dv] + (z^5uv^2 - u^2z^5v^3)dz = 0$$

$$\text{or } z^6[(vu^2 - v^2)du + (u^2v^2 - u^3)dv] + z^5[vu^3 - v^2u + u^2v^3 - u^3v + uv^2 - u^2v^3]dz = 0$$

Obviously the coefficient of dz is zero and the resulting equation

$$(vu^2 - v^2)\,du + (u^2v^2 - u^3)du = 0$$

involves only two variables u and v. To solve it, we divide throughout by $u^2v^2$ and get

$$\left(\frac{1}{v} - \frac{1}{u^2}\right)du + \left(1 - \frac{u}{v^2}\right)dv = 0$$

or
$$\frac{1}{v}\,du - \frac{u}{v^2}\,dv - \frac{1}{u^2}\,du + dv = 0$$

or
$$\frac{vdu - udv}{v^2} - \frac{1}{u^2}\,du + dv = 0$$

Integrating, we obtain

$$\frac{u}{v} + \frac{1}{u} + v = c$$

Replacing u by $\frac{x}{z}$ and v by $\frac{y}{z}$, we have

$$\frac{x}{y} + \frac{z}{x} + \frac{y}{z} = c$$

as the required solution.

**NOTE** Since $xP + yQ + zR = yz^2x^3 - xy^2z^3 + zx^2y^3 - yz^2x^3 + xy^2z^3 - zx^2y^3 = 0$

so $\dfrac{1}{xP + yQ + zR}$ will not be an integrating factor of given equation.

## EXERCISE 9.2

If integrable, solve

**1.** $y^2dx - z\,dy + y\,dz = 0$ **2.** $x\,dx + z\,dy + (y + 2z)dz = 0$

**3.** $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$

**4.** $(y + z)z\,dx - zx\,dy + xy\,dz = 0$

**5.** $2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0$

**6.** $yz\,dx + (xz - yz^3)dy - 2xy\,dz = 0$ **7.** $2yz\,dx + zx\,dy - xy(1 + z)dz = 0$

**8.** $z\,dx + zx\cos y\,dy + x(1 - z)(\log x + \sin y)\,dz = 0$

**9.** $yz\log z\,dx - zx\log z\,dy + xy\,dz = 0$

**10.** $(yz + 2x)dx + (zx + 2y)dy + (xy + 2z)dz = 0$

**11.** $z^2dx + (z^2 - 2yz)\,dy + (2y^2 - yz - zx)dz = 0$

**12.** $(2x^3 - z)z\,dx + 2x^2yz\,dy + x(z + x)dz = 0$

**13.** $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$

**14.** $(2x^2 + 2xy + 2x\,z^2 + 1)\,dx + dy + 2z\,dz = 0$

**15.** $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$

**16.** $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$

## 9.3 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

In this section we shall study the solutions of simultaneous equations of the first order and first degree. We shall restrict ourselves to the consideration of equations involving only three variables. A general form of such equations is written as

$$P_1\,dx + Q_1\,dy + R_1\,dz = 0 \quad \ldots\ldots (1)$$

$$P_2\,dx + Q_2\,dy + R_2\,dz = 0 \quad \ldots\ldots (2)$$

where the coefficients $P_1, Q_1, R_1, P_2, Q_2, R_2$ are all functions of x, y, z.

Let the equations (1) and (2) have

$$u(x, y, z) = c_1 \text{ and } v(x, y, z) = c_2$$

as their independent solutions respectively. Then these two solutions taken together constitute the general solution of the above system of equations.

On the other hand, if one or both of the given equations is not integrable, then it is preferable to express them in the differential form as

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

i.e., $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots\ldots (3)$$

where P, Q, R are functions of x, y, z. If the need be we can increase the number of equations with the help of algebra. The equations (3) may be solved by the following methods.

### 9.3.1 I. METHOD OF GROUPING

Suppose that one of the variables, say y, is either absent or can be cancelled from any two fractions of equations (3). Then we shall get an integral of the type

$$\phi_1(x, z) = c_1 \qquad \ldots\ldots (4)$$

Similarly, by choosing any other two fractions of (3) in which one variable, say x, is absent or can be cancelled we shall get a solution of the form

$$\phi_2(y, z) = c_2 \qquad \ldots\ldots (5)$$

The two independent solutions (4) and (5) taken together constitute the complete solution of the given system.

Sometimes it so happens that after finding one integral, say (4), the other integral cannot be obtained by the same rule. Then we make use of the known integral given by (4) to remove x and get an equation in terms of other two variables y, z and hence find the second integral. The constant of integration of first integral now involved in the second integral, is eliminated later on.

**EXAMPLE 1** Solve $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$

**SOLUTION** Taking the first two fractions and cancelling z, we have

$$\frac{dx}{y^2} = \frac{dy}{x^2} \Rightarrow x^2dx = y^2dy$$

which is an equation in x and y only and on integration gives

$$x^3 - y^3 = c_1 \qquad \ldots\ldots (1)$$

Similarly by taking the first and third fractions and cancelling $y^2$, we get

$$\frac{dx}{z} = \frac{dz}{x} \Rightarrow xdx - zdz = 0$$

Its solution is $x^2 - z^2 = c_2$ ..... (2)

Thus (1) and (2) taken together constitute the general solution of the given equations.

**EXAMPLE 2** Solve $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2}$

**SOLUTION** From the first two fractions, we get

$$dx + dy = 0 \Rightarrow x + y = c_1 \quad \ldots\ldots (1)$$

Taking the first and the third fractions and making use of (1), we have

$$\frac{dx}{z} = \frac{dz}{z^2 + c_1^2} \Rightarrow dx = \frac{zdz}{z^2 + c_1^2}$$

Its integral is $x = \frac{1}{2}\log(z^2 + c_1^2) + c$

or $2x = \log(z^2 + c_1^2) + c_2$

$= \log[z^2 + (x + y)^2] + c_2$ ..... by (1)

Thus the complete solution of the given system is

$$x + y = c_1 \text{ and } 2x = \log[z^2 + (x + y)^2] + c_2.$$

### 9.3.2 II. METHOD OF MULTIPLIERS

We know from algebra that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \quad \ldots\ldots (1)$$

Now by choosing suitable multipliers l, m, n which may be constants or functions of x, y, z we make

$$lP + mQ + nR = 0$$

Consequently, $ldx + mdy + ndz = 0$

Let $f(x, y, z) = c_1$ be the solution of this equation. Similarly we choose another set of multipliers $l', m', n'$ such that

$$l'P + m'Q + n'R = 0$$

Then $$l'dx + m'dy + n'dz = 0$$

Integrating this equation we get another solution, say

$$g(x, y, z) = c_2$$

The two solutions $f(x, y, z) = c_1$ and $g(x, y, z) = c_2$ taken together constitute the complete solution of the given simultaneous equations.

Sometimes only one integral can be obtained by the use of multipliers. In such cases we will combine the two methods.

If the numerator of $\dfrac{ldx + mdy + ndz}{lP + mQ + nR}$ ..... (2)

is exact differential of the non-zero denominator then (2) can be combined with a suitable fraction already existing in the given equation to get the integral. However, in some problems we can choose another set of multipliers such that the numerator of

$$\frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R} \quad \ldots\ldots \quad (3)$$

is exact differential of non-zero denominator. Then the combination of the fractions (2) and (3) is found more convenient for an integral of the equation.

**EXAMPLE 1** Solve $\dfrac{adx}{(b - c)yz} = \dfrac{bdy}{(c - a)zx} = \dfrac{cdz}{(a - b)xy}$

**SOLUTION** Here a set of multipliers which when multiplied by P, Q, R give the sum zero, are x, y, z.

$$\therefore \quad \text{each given fraction} = \frac{axdx + bydy + czdz}{0}$$

$$\therefore \quad axdx + bydy + czdz = 0$$

Integrating we get

$$ax^2 + by^2 + cz^2 = 2C = c_1 \quad \ldots\ldots \quad (1)$$

Similarly using $ax$, by, cz as multipliers, we have

$$\text{each given fraction} = \frac{a^2x\,dex + b^2y\,dy + c^2z\,dz}{0}$$

$$\therefore \quad a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$$

Integration gives

$$a^2x^2 + b^2y^2 + c^2z^2 = 2c = c_2 \qquad \ldots\ldots (2)$$

(1) and (2) taken together constitute the complete solution of the given equations.

**EXAMPLE 2** Solve $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2zx}$

**SOLUTION** From the last two fractions, cancelling 2x we have

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_1$$

$$\Rightarrow \quad y = c_1z$$

Now using x, y, z as multipliers, we have

$$\text{each of given fraction} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\therefore \quad \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dy}{2xy}$$

$$\text{or} \quad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = \frac{dy}{y}$$

Integrating, we obtain

$$\log(x^2 + y^2 + z^2) = \log y + \log c_2$$

$$\text{i.e.,} \quad x^2 + y^2 + z^2 = c_2y$$

Thus the required solution consists of

$$y = c_1z, \quad x^2 + y^2 + z^2 = c_2y$$

**EXAMPLE 3** Solve $\dfrac{dx}{y + z} = \dfrac{dy}{z + x} = \dfrac{dz}{x + y}$

**SOLUTION** Taking the multipliers 1, −1, 0 ; 0, 1, −1 and −1, 0, 1 respectively, we have each member of the given equation equal to

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dz - dx}{x - z} \qquad \ldots\ldots (1)$$

Considering the first two terms of (1), we have

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

Integrating $\log(x - y) = \log(y - z) + \log c_1$

or $x - y = (y - z)c_1$ ..... (2)

Similarly the last two terms of (1) give

$$y - z = (z - x)c_2 \quad \text{..... (3)}$$

Equations (2) and (3) constitute the simultaneous solution of the given equations.

## EXERCISE 9.3

Solve

1. $dx + 2dy - (x + 2y)dz = 0,$
$2dx + dy + (x - y)dz = 0.$

2. $x\,dx + y\,dy + z\,dz = 0,$
$yz\,dx + zx\,dy + xy\,dz = 0.$

3. $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

4. $\frac{dx}{x} = \frac{dy}{x + z} = \frac{dz}{-z}$

5. $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$

6. $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$

7. $\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{y - x}$

8. $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

9. $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$

10. $\frac{xdx}{z^2 - 2yz - y^2} = \frac{dy}{y + z} = \frac{dz}{y - z}$

11. $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$

12. $\frac{dx}{1 + y} = \frac{dy}{1 + x} = \frac{dz}{z}$

13. $\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$

14. $\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$

15. $\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$

16. $\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x + y)z}$

## 9.4 PARTIAL DIFFERENTIAL EQUATIONS

We know that an equation involving partial differential coefficients is called a partial differential equation. The order of a partial differential equation is the order of the highest order differential coefficient occuring in it and its degree is the degree of this derivative.

If $z = f(x, y)$, where x and y are independent variables, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ will be denoted by p and q respectively. The notation used for the second order derivatives is :

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad t = \frac{\partial^2 z}{\partial y^2}$$

### 9.4.1 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

The partial differential equations are formed either by the elimination of arbitrary constants from a relation between three or more variables x, y, z, ... or by the elimination of arbitrary functions of these variables. We shall illustrate the method by the following solved examples.

**EXAMPLE 1** Form a partial differential equation by eliminating the arbitrary constants $a$,b from

$$z = (x^2 + a^2)(y^2 + b^2) \quad \ldots\ldots (1)$$

**SOLUTION** Differentiating (1) partially w.r.t.x and y, we get

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b^2) \quad \ldots\ldots (2)$$

and

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a^2) \quad \ldots\ldots (3)$$

Multiplying (2) and (3), we have

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 4xy(x^2 + a^2)(y^2 + b^2)$$

or

$$pq = 4xyz$$

which is a partial differential equation of first order.

**NOTE** In the above example, we have two independent variables x, y and two arbitrary constants and the elimination of these constants resulted in a partial differential equation of first order. If the number of constants is greater than the number of independent variables, then the elimination of the constants will give rise to a partial differential equation of order higher than first.

**EXAMPLE 2** Form a partial differential equation by eliminating the function f from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \ldots\ldots (1)$$

**SOLUTION** Differentiating (1) partially w.r.t.x, we get

$$\frac{\partial z}{\partial x} = p = 2f'\left(\frac{1}{x} + \log y\right).\left(-\frac{1}{x^2}\right)$$

or $$-px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \ldots\ldots (2)$$

Again differentiating (1) partially w.r.t.y, we have

$$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right)$$

or $$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \ldots\ldots (3)$$

From (2) and (3) by eliminating f′, we obtain

$$-px^2 = qy - 2y^2$$

or $$x^2p + yq = 2y^2$$

which is a partial differential equation of first order and first degree.

**NOTE** Thus we observe that from a given relation involving x , y , z we obtain a P.D.E. of first order on eliminating one arbitrary function. If there are more than one arbitrary function, then the resulting P.D.E. will be, in general, of higher order.

## EXERCISE 9.4(a)

Find the partial differential equation by eliminating the arbitrary constants/functions from

1. $z = ax + by + ab$
2. $z = (x + a)(y + b)$
3. $z = f(x^2 - y^2)$
4. $z = f\left(\frac{y}{x}\right)$
5. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
6. $x^2 + y^2 + (z - c)^2 = a^2$
7. $x + y + z = f(x^2 + y^2 + z^2)$
8. $y = f(x - 3t) + F(x + 3t)$
9. $z = x + y + f(xy)$
10. $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

### 9.4.2 SOLUTION OF A P.D.E.

The solution of a partial differential equation is a function of independent variables which satisfies the differential equation.

Let us first consider those partial differential equations which can be solved by direct integration. In place of the usual constants of integration we use arbitrary functions of the variables kept fixed.

Consider,

$$\frac{\partial z}{\partial x} = y \cos x \qquad \ldots\ldots (1)$$

Integrating w.r.t.x, holding y fixed, we have

$$z(x, y) = \int y \cos x \, \partial x + f(y)$$

$$= y \sin x + f(y) \qquad \ldots\ldots (2)$$

where f is an arbitrary function rather than an arbitrary constant. By differentiation we can verify that for any f, (2) is a solution of (1), so that it is the general solution of the equation.

As an example of a simple second order equation, consider

$$\frac{\partial^2 z}{\partial x^2} = 2x$$

Integrating w.r.t.x, keeping y fixed, we get

$$\frac{\partial z}{\partial x} = x^2 + f(y)$$

Integrating again, we have

$$z(x, y) = \frac{x^3}{3} + x\, f(y) + g(y)$$

Similarly, if we consider $\frac{\partial^2 u}{\partial x^2} = 1$, u being a function of there variables x, y, z then proceeding as above its general solution will be

$$u(x, y, z) = \frac{x^2}{2} + x\, f(y, z) + g(y, z)$$

where f and g are arbitrary functions of two variables.

From the above consideration we see that the number of independent arbitrary functions in the general solution of a P.D.E is the same as the order of the equation.

### 9.4.3 LAGRANGE'S LINEAR PARTIAL DIFFERENTIAL EQUATION

The linear first order partial differential equation of the form

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \quad \text{i.e.,} \quad Pp + Qq = R,$$

where P, Q, R are functions of x, y, z is known as Lagrange's Linear Partial Differential Equation.

The procedure for solving this equation will be explained by the following theorem.

### 9.4.4 THEOREM

The general solution of the linear partial differential equation

$$Pp + Qq = R$$

is

$$f(u, v) = 0 \qquad \ldots\ldots (1)$$

where f is an arbitrary function and $\ldots\ldots (2)$

$$u(x, y, z) = c_1, \text{ and } v(x, y, z) = c_2$$

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

in which P, Q, R are functions of x, y, z.

**PROOF** We shall first show that the differential equation (1) is obtained by eliminating the arbitrary function f(u , v) from the equation (2).

Differentiating (2) partially w.r.t.x, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad \ldots\ldots (3)$$

If z is taken as dependent and x, y as independent variables,

then $$\frac{\partial z}{\partial x} = p\,,\ \frac{\partial z}{\partial y} = q\,,\ \frac{\partial y}{\partial x} = 0\,,\ \frac{\partial x}{\partial y} = 0$$

Hence (3) reduces to

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}\right) = 0$$

or $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z}\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}\right) \quad \ldots\ldots (4)$$

Similarly, differentiating (2) partially w.r.t. y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z}\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\,\frac{\partial v}{\partial z}\right) \quad \ldots\ldots (5)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from equations (4) and (5), we obtain

$$\left[\frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z}\right]\left[\frac{\partial v}{\partial y} + q\,\frac{\partial v}{\partial z}\right] = \left[\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z}\right]\left[\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}\right]$$

Re–arranging the terms, we have

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p + \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

or $$Pp + Qq = R \quad \ldots\ldots (6)$$

where $$P = \frac{\partial(u, v)}{\partial(y, z)} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$Q = \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v \partial u}{\partial z \partial x}$$

$$R = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

Hence (2) is the general solution of (6) which is a P.D.E. of first order and first degree in p and q.

We are now required to find the values of u and v and consequently the function f(u , v). Let us assume that

$$u(x, y, z) = c_1 \quad \ldots\ldots (7)$$

and

$$v(x, y, z) = c_2 \quad \ldots\ldots (8)$$

are two independent solutions of a certain differential equation which we are interested to find. Taking differentials of (7) and (8), we obtain

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \ldots\ldots (9)$$

and

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \ldots\ldots (10)$$

Solving (9) and (10) for dx , dy , dz we get

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}$$

or

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots\ldots (11)$$

Hence we have established that $u = c_1$ and $v = c_2$ are the solutions of (11), the equation we are in search of. After having determined u and v we substitute them in $f(u , v) = 0$ to obtain the required solution of (1)

**NOTE** Equations (11) are know as Lagrange's auxiliary equations for (1)

## WORKING PROCEDURE

### FIRST STEP

Write the given P.D.E. in the standard form of Lagrange's equation

$$\text{i.e., } Pp + Qq = R$$

### SECOND STEP

Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

### THIRD STEP

Find two independent solutions $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ of the auxiliary equations.

### FOURTH STEP

Then $f(u, v) = 0$ or $u = \phi(v)$ or $v = \phi(u)$

is the required solution of $Pp + Qq = R$.

**EXAMPLE 1** Find the general solution of

$$yq - xp = z$$

**SOLUTION** We can write the given equation in standard form as

$$-xp + yq = z$$

The subsidiary equations are

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

From the first and second fractions, we have

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integration gives $\log x + \log y = \log a$

or $\log xy = \log a \Rightarrow xy = a$ ..... (1)

From the second and third members, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating $\log y = \log z + \log b$

or $\log \frac{y}{z} = \log b \Rightarrow \frac{y}{z} = b$ ..... (2)

From (1) and (2) the general solution of given equation is

$$f\left(xy, \frac{y}{z}\right) = 0$$

The solution may also be written as

$$xy = f\left(\frac{y}{z}\right) \quad \text{or} \quad \frac{y}{z} = f(xy)$$

**EXAMPLE 2** Solve $x^2p + y^2q = (x + y)z$

**SOLUTION** The subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z} \quad \text{..... (1)}$$

Taking the first and second fractions and integrating,

$$-\frac{1}{x} = -\frac{1}{y} + a \Rightarrow \frac{1}{y} - \frac{1}{x} = a \quad \text{..... (2)}$$

Equations (1) can also be written as

$$\frac{\frac{dx}{x}}{x} = \frac{\frac{dy}{y}}{y} = \frac{\frac{dz}{z}}{x + y} = \frac{\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z}}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$$

Integrating $\log x + \log y - \log z = \log b$

or $\log \frac{xy}{z} = \log b \Rightarrow \frac{xy}{z} = b$ ..... (3)

From (2) and (3), we get

$$\frac{1}{y} - \frac{1}{x} = f\left(\frac{xy}{z}\right)$$

**EXAMPLE 3** Solve $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

**SOLUTION** Re–arranging the given equation to the standard form of Lagrange's linear equation, we have

$$x(z-2y^2)\,p + y(z-y^2-2x^3)\,q = z(z-y^2-2x^3)$$

Its auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \ldots\ldots (1)$$

From the last two members of (1) we get

$$\frac{dy}{y} = \frac{dz}{z}$$

which gives on integration

$$\log y = \log z + \log a \Rightarrow y = az \quad \ldots\ldots (2)$$

From the first and third fractions of (1) we have

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}$$

Putting $y = az$ from (2), we obtain

$$\frac{dx}{x(z-2a^2z^2)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

or $$\frac{dx}{x(1-2a^2z)} = \frac{dz}{z-a^2z^2-2x^3}$$

or $$zdx - a^2z^2dx - 2x^3dx = xdz - 2a^2xz\,dz$$

or $$(xdz - zdx) - a^2(2xzdz - z^2dx) + 2x^3dx = 0$$

or $$\frac{xdz - zdx}{x^2} - a^2\left[\frac{2xzdz - z^2dx}{x^2}\right] + 2xdx = 0$$

Integration gives

$$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = b \quad \ldots\ldots (3)$$

From the equations (2) and (3), we have

$$\frac{y}{z} = f\left(\frac{z}{x} - \frac{a^2z^2}{x} + x^2\right)$$

as the required solution of given equation.

## EXERCISE 9.4(b)

Solve

1. $xp + yq = 3z$
2. $p \tan x + q \tan y = \tan z$
3. $zp + yq = x$
4. $y\,zp + x\,zq = xy$
5. $y^2p - x\,yq = x(z - 2y)$
6. $(y - z)p + (x - y)q = z - x$
7. $(y + z)p - (x + z)q = x - y$
8. $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$
9. $\frac{y - z}{yz}p + \frac{z - x}{zx}q = \frac{x - y}{xy}$
10. $(y + z)p + (z + x)q = x + y$
11. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$
12. $(p - q)z = z^2 + (x + y)^2$
13. $(xp - yq)z = y^2 - x^2$
14. $(x + 2z)p + (4zx - y)q = 2x^2 + y$
15. $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$
16. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$
17. $(y + zx)p - (x + yz)q = x^2 - y^2$

# ANSWERS

## EXERCISE 1.1

1. (i) $2i$ (ii) $-2i$ (iii) $-1$

   (iv) $\frac{a^4 + b^4 - 6a^2b^2}{(a^2 + b^2)} + \frac{4iab\,(a^2 - b^2)}{(a^2 + b^2)}$ (v) $\frac{-2}{25} + \frac{11i}{25}$

2. (i) $\text{Cis}\,(2\alpha + \beta)$ (ii) $2\cos\alpha\,[\cos 3\alpha - i \sin 3\alpha]$ (iii) $i \tan\theta$

3. (i) $\frac{1}{2} + \frac{1}{2}\sqrt{3}\,i$ (ii) $\frac{1}{2} + \frac{i}{2}, \frac{1}{\sqrt{2}}; \frac{\pi}{4}$ (iii) $1, \frac{\pi}{2}$

5. (i) $1, \pi; 1, 0$ (ii) $\sqrt{2}, -\frac{\pi}{4}; \sqrt{5}, -\tan^{-1}\frac{1}{2}$

6. $x^2 + y^2 = y^2g^2 + p^2 + x^2q^2 - 2pqx$ 8. $\frac{\pi}{2}, \sqrt{13}$

9. $1 + 18i, 3 + 4i$ 11. $-6 + 9i$

## EXERCISE 1.2

14. 1

## EXERCISE 1.3

1. $\text{Cis}\,(28\theta)$

2. $2^8 \sin^8 \theta/2\,[\text{Cis}\,4\theta]$

3. $2^8 \cos^8 \theta/2\,[\text{Cis}\,5\theta]$

4. $\frac{4\pi}{3}$, $k = 0, \pm 1, \pm 2, \ldots$

10. (i) $\text{Cis}\,\frac{(2k + 1)}{7}\pi$, where $k = 0, 1, 2, 3, 4, 5, 6$

    (ii) $-1, \frac{1 \pm i}{\sqrt{2}}, -\frac{-1 \pm i}{\sqrt{2}}, \frac{1 \pm i\sqrt{3}}{\sqrt{2}}$

    (iii) $\text{Cis}\left(\frac{2k\pi}{12}\right)$, where $k = 0, 1, 2, 3, 4, \ldots, 11$

## EXERCISE 1.6

3. (i) $\log_e^2 + i(\pi + 2k\pi)$, where k is any integer.

(ii) $\log_e^2 + i\left(\frac{\pi}{3} + 2k\pi\right)$, where k is any integer.

(iii) $1 + 2k\pi i$, where k is any integer.

4. $e^{2x}$; $e^{-2xy}$

## EXERCISE 1.7

12. (i) $\frac{\pi}{2} - 4i$ (ii) $\frac{1}{i}\log_e(2 + \sqrt{3})$

## EXERCISE 1.8

2. $e^{-\pi/2}$ 12. (i) $\frac{1}{2}\tan^{-1}(4/5)$ (ii) 0 (iii) Null set.

## EXERCISE 1.9

3. (i) $C = \cos\dfrac{[-\cos^n\theta\cos(n+1)\theta + \cos^{n+1}\theta\cos n\theta]}{\sin^2\theta}$

(ii) $S = \cos\theta\dfrac{[\sin\theta - \cos^n\theta\sin(n+1)\theta + \cos^{n+1}\theta\sin n\theta]}{\sin^2\theta}$

4. (i) $\cot^2\theta\left[\dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin(n+1)\theta}{\cos^{n+1}\theta} + \dfrac{\sin n\theta}{\cos^{n+2}\theta}\right]$

(ii) $\cot^2\theta\left[1 - \sec^2\theta - \dfrac{\cos(n+1)\theta}{\cos^{n+1}\theta} + \dfrac{\cos n\theta}{\cos^{n+2}\theta}\right]$

5. $2^n\cos^n x/2\left[\cos\dfrac{nx}{2}\right]$

6. $\dfrac{1 - C^{n+1}\cos(n+1)x - C\cos x + C^{n+2}\cos nx}{1 + C^2 - 2C\cos x}$

7. $\dfrac{\sin\theta + (2n+3)\sin n\theta - (2n+1)\sin(n+1)\theta}{2(1-\cos\theta)}$

8. $\dfrac{\sin^2\theta}{1-\sin 2\theta+\sin^2\theta}$

10. $\sqrt{2\cos\theta}\cos\theta/2$

11. $\sqrt{2\cos\theta}\sin\theta/2$

12. $\dfrac{1}{\sqrt{2\cos\theta/2}}\cos\dfrac{\theta}{4}$

13. $\dfrac{1}{\sqrt{2\cos\theta/2}}\sin\theta/4$

14. $\cos(\theta-\phi)\sin(\cos\phi)\cosh(\sin\phi)$

$-\sin(\theta-\phi)\cos(\cos\phi)\sinh(\sin\phi)$

15. $e^{x\cos\alpha}\cos(x\sin\alpha)$

16. $\frac{1}{2}\log(1-2x\cos\alpha+x^2), |x|<1$

17. $x\cos x-\sin x\log_e(2\cos x)$

18. $e^{\cos^2\theta}[\cos(\cos\theta+\sin\theta+\theta)]$

20. $\tan^{-1}\left[\dfrac{x\sin\theta}{1+x\cos\theta}\right]$

21. $\log(2\cos\theta/2)$

22. $e^{\cos\theta}[\sin(\sin\theta)]$

23. $\cos(\cos\theta)\cosh(\sin\theta)$

24. $\dfrac{4\sin\theta}{5-4\cos\theta}$

25. $\frac{1}{4}\log\left(\dfrac{1+c^2+2c\cos\theta}{1+c^2-2c\cos\theta}\right)$

## EXERCISE 2.1

1. (i) yes (ii) yes (iii) yes

2. All the permutations of A.

3. $f(x) = a, \varphi(x) = b, x = 1, 2, 3.$

6. (i) not onto (ii) not (1 – 1).

7. (1 – 1) – correspondence.

8. (i) $9x^2-6x-2$ (ii) $3x^2-10$ (iii) 2 (iv) –2

9. (i) $(x-1)^2$ (ii) $x^2-2x$ (iii) $(x^2-1)^2$ (iv) $x-2$ (v) $x^4$

10. No

## EXERCISE 2.2

1. (i) true (ii) true (iii) true (iv) commutative but not cancellative
(v) false (vi) false (vii) false (viii) true
(ix) not cancellative but observes the identity property.

## EXERCISE 2.3

1. (vii) No 6. (i) 2, 3 (ii) $b, ba^2, ba$.

11. All the non-identity elements.

## EXERCISE 2.4

1. (i) yes (ii) yes (iii) yes (iv) yes

2. (i) yes (ii) (a) yes (b) yes (iii) yes (iv) yes
   (v) yes (vi) yes

3. yes 4. yes

## EXERCISE 2.5

1. (i) yes (ii) yes (iii) yes (iv) yes (v) yes
   (vi) yes (vii) yes

6. If $C_4 = \langle x \rangle$ then $\langle x^2 \rangle$ is the only proper subgroup of $C_4$.

7. $\langle x^2 \rangle$, $\langle x^4 \rangle$ are the only proper subgroups of $C_8$.

## EXERCISE 3.1

1. (i) (a) field (ii) not a ring (iii) not a ring (iv) a field (v) a field
   (vi) not a ring (vii) not a ring (viii) ring with identity (ix) field
   (x) field (xi) field (xii) field
   (xiii) commutative ring with identity

2. (i) yes; $m \neq 0, \frac{1}{m} \notin Z$ (ii) yes (iii) No. (iv) yes
   (vi) No (vii) No (viii) No. (ix) No (x) No.

3. yes 4. No.

## EXERCISE 3.2

3. yes 4. yes 5. No.

## EXERCISE 3.3

1. (a) (19, 10) (b) (27, 20) (c) (15, 14, 1) (d) (–1, 4, 10, 34)

2. No. 3. $0(1, 2, 3) + \frac{7}{3}(0, 1, 2) + \frac{8}{3}(3, 1, -1)$ 4. yes

6. (a) L.D (b) L.I (c) L.I (d) L.D (e) L.I
(f) L.D

11. $\lambda = 4$

## EXERCISE 3.4

1. (a) yes (b) No (c) yes (d) No

4. For each $m \in R$, $\{mr : r \in R\}$ forms a subspace of R.

5. (a) yes (b) yes (c) yes.

7. (a) yes (b) No (c) No (d) No

8. (a) yes (b) yes

## EXERCISE 3.5

6. (i) No (ii) yes (iii) yes 8.(i) $\{v_1, v_2, v_3, v_4\}$ (ii) a = 1

9. $\{v_1, v_2, v_3\}$ 10. $\{1, x, x^2, x^3, \cdots, x^n, \cdots\}$

12. yes

13. (i) 2 (ii) 2 (iii) 1

## EXERCISE 3.6

1. (i) Linear (ii) Linear (iii) Linear (iv) Linear (v) Linear
(vi) not linear (vii) not linear

2. (i) linear (ii) linear (iii) not linear (iv) not linear (v) linear

3. (i) linear (ii) linear (iii) linear (iv) not linear (v) linear

10. No.

## EXERCISE 4.1

**1.** (i) $\frac{1}{9}\begin{bmatrix} 4 & 22 \\ -22 & 41 \end{bmatrix}$ (ii) $\begin{bmatrix} -109 & 82 \\ 89 & 86 \end{bmatrix}$

**3.** $x \in \{0, -2\}$

**5.** The diagonal entries in $D^n$ are the nth powers of the corresponding entries in D.

**6.** (i) $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

**10.** (i) 3 (ii) not nilpotent (iii) 2.

**11.** periodic of period 4.

**12.** (i) Idempotent (ii) Idempotent.

**13.** (i) Involutory (ii) Involutory.

**16.** (i) $B = \begin{bmatrix} 5/3 & -2 & 4/3 \\ -2 & 3 & -2 \\ 4/3 & 2 & 5/3 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & -1 & 3 \\ 4 & 4 & 4 \\ -5 & 1 & 1 \\ 4 & 4 & 4 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \end{bmatrix}$

**20.** (iii) $P = \frac{1}{2}\begin{bmatrix} 8 & 1 & 1+4i \\ 1 & 0 & 5-i \\ 1-4i & 5+i & 14 \end{bmatrix}$, $Q = \frac{1}{2}\begin{bmatrix} 6i & -1 & -1+2i \\ 1 & 2i & 7+i \\ 1+2i & -7+i & 0 \end{bmatrix}$

**21.** (i) $b = 0$

**24.** $c = \pm\frac{1}{\sqrt{3}}$, $b = \pm\frac{1}{\sqrt{6}}$, $a = \pm\frac{1}{\sqrt{2}}$

**30.** (i) $\begin{bmatrix} 3 & 3/2 & 0 \\ 3/2 & 2 & 9/2 \\ 0 & 9/2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -7/2 & 1 \\ 7/2 & 0 & -3/2 \\ -1 & 3/2 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 7 & 0 \\ 7 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$

**32.** $a = 1$, $b = -1$

## EXERCISE 4.2

**1.** (i), (iii), (iv) and (vi) are in their echelon form with pivots 2,3; 1,1,1 ;1,2; 1,1 respectively.

**2.** (iii) and (vi)

**5.** (i) (a) 3 (b) 2 (c) 2 (d) 4 (e) 3 (f) 3 (g) 2.

**6.** (a) $\frac{1}{19}\begin{bmatrix} 9 & -4 & 1 \\ 1 & 8 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ (b) no inverse (c) $\frac{1}{39}\begin{bmatrix} 7 & 3 & 22 & 27 \\ -1 & -6 & -31 & -15 \\ 8 & 9 & 14 & 3 \\ -5 & 9 & 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & +2 & -1 & -1 \\ 1 & 5 & -2 & -2 \\ 0 & 4 & -2 & -1 \\ 0 & -2 & 1 & 1 \end{bmatrix}$

**7.** (i) $\begin{bmatrix} 0 & 3 & 0 \\ 4 & 2 & 2 \\ 4 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

**8.** (a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -14 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

**9.** (a) $\begin{bmatrix} 0 & 1/4 & 1/2 \\ -i & \frac{3i}{4} & \frac{1}{2} \\ 0 & 1/4 & 1/2 \end{bmatrix}$, (b) $\begin{bmatrix} -i/2 & 1/2 & 0 \\ -\frac{3i}{2} & 1/2 & -1 \\ -i & 1 & -1 \end{bmatrix}$

**10.** (i) $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 3 & 2 & -6 \\ 1 & -6 & 3 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} -4 & 7 & 1 \\ 3 & -3 & 5 \end{bmatrix}$

**11.** $a = 1$, $b = -i$

## EXERCISE 4.3

**1.** (i) (1, 1, 2) (ii) (2, −1, 2)

**2.** (i) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 5 & 0 \\ 1 & -1 & 1 \\ -5 & 1 & 0 \end{bmatrix}$

**3.** $\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$

**4.** $\begin{bmatrix} -1 & 0 & -12/8 \\ 1 & 14/8 & 7/8 \end{bmatrix}$

**5.** $\begin{bmatrix} 2 & 3 & 6 \\ -2 & 1 & -1 \end{bmatrix}$

**6.** $\begin{bmatrix} 1 & 3 \\ 7 & 6 \\ -5 & -10 \end{bmatrix}$

**8.** $T(u_1, u_2) = (2u_2, u_1 + u_2, 2u_1)$

**9.** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

**8.** $T(u_1, u_2, u_3) = (u_1, 3u_1 + u_3)$

**11.** $\begin{bmatrix} 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**12.** $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

# EXERCISE 4.4

1 (i) $(x_1, x_2, x_3) = (2, -1, 2)$ (ii) $(x_1, x_2) = \left(\frac{-1}{3}, 2\right)$

(iii) $(x_1, x_2, x_3, x_4) = (23 - 8c, -10 + 4c, 4 - 2c, c)$

(iv) Infinite solutions. (v) $(x_1, x_2, x_3, x_4) = (1 - \lambda, \lambda, -1, 2), \lambda \in R$.

2. (i) $x = (-12/5, 18/5, -1/10)$ (ii) $x = \left(\frac{1}{2}, 0, 0\right)$

(iii) $x = (-4, -2, 2)$ (iv) $x = \left(-\frac{1}{3}, -\frac{2}{3}, 1\right)$

(v) $x = \left(\frac{7}{13}, \frac{4}{13}, \frac{2}{13}\right)$

3. (i) $x = \left(\frac{13}{5} + \frac{4}{5}c, \frac{4}{5} - \frac{3}{5}c, c\right), c \in R$.

(ii) $x = (3 - 2x_3, -1 - x_3, x_3), x_3 \in R$.

(iii) $x = (11 + c, 5 + c, 3 + c, c), c \in R$.

(iv) $x = (1, 1/2, -1, 3/2, 1/2)$

(v) $x = (4x_3 - x_5 - 11, -x_3 + 2x_5 + 2, x_3, 3x_5, x_5); x_3, x_5 \in R$.

4. (i) $\alpha = 7$ (ii) $\alpha = -1$

5. (i) when $\alpha = \pm 1$, no solution, otherwise the system has a unique solution $x = \frac{1}{1 - \alpha^2}(1 - 4\alpha, 4 - \alpha)$

(ii) when $\alpha = 1$, no solution, otherwise the system has a unique solution $x = \frac{1}{2 - 2\alpha}(-8, 5 - \alpha)$

(iii) when $\alpha = 2$, $\alpha = 0$ no solution. Otherwise the system has a unique solution

(iv) No solution for $\alpha = 0$, for $\alpha = -1$, infinite solution.
For $\alpha \neq 1$ zero solution.

## EXERCISE 4.5

**1.** (i) 206 (ii) −6 (iii) 5/9 (iv) −3 (v) 60

**2.** $|A| = 91$ **3.** $\lambda = 1$

**4.** $x = 1, 2$ **6.** (i) 124 (ii) −154

**7.** (i) −1014 (ii) −189 (iii) 17 (iv) 3

**9.** (i) 32 (ii) 250 (iii) −2 (iv) $\frac{1}{2}$ (v) 2

**12.** $a = -32$

**14.** (i) $\frac{1}{22}\begin{bmatrix} 0 & 2 & 8 \\ 22 & -12 & -26 \\ 0 & 3 & 1 \end{bmatrix}$ (ii) $\frac{-1}{22}\begin{bmatrix} 7 & -2 & -23 \\ 7 & -2 & -1 \\ -11 & 0 & 11 \end{bmatrix}$

(iii) $\frac{1}{2}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & -1 & 3 \end{bmatrix}$ (vi) $\frac{1}{17}\begin{bmatrix} 7 & 2 & -5 \\ -18 & 7 & 8 \\ -5 & 1 & 6 \end{bmatrix}$

## EXERCISE 5.1

**1.** (i) Does not exist (ii) 0, (iii) Does not exist (iv) 6,
(v) $\log_e(4/3)$ (vi) 0, (vii) 1

**2.** 144

## EXERCISE 5.3

**1.** 5/2 **2.** Dgt. **3.** Dgt. **4.** 10,000 **5.** Dgt. **6.** $1 + \sqrt{5}$

**7.** $\frac{1}{2}$ **8.** $\frac{17}{6}$ **9.** 2/3 **10.** Dgt. **11.** Dgt. **13.** Dgt. 15. Dgt.

## EXERCISE 5.4

**1.** Cgt. **2.** Dgt. **3.** Dgt. **4.** Dgt. **5.** Dgt. **6.** Cgt.

**7.** Cgt. **8.** Cgt. **9.** Cgt. **10.** Cgt. **11.** Cgt. **13.** Cgt.

**14.** Dgt. **15.** Dgt. **16.** Cgt. **20.** Cgt. **21.** Cgt. **22.** Dgt.

23 Dgt. 24. Cgt. 25. Cgt. 27. Cgt. 29. Cgt. 30. Dgt.

31. Cgt. 33. Dgt. 34. Cgt.

## EXERCISE 5.5

1. Cgt. 2. Cgt. 3. Dgt. 4. Cgt. 5. Cgt. 6. Dgt.

7. Cgt. 8. Dgt. 9. Dgt. 10. Cgt. 11. Cgt. 12. Cgt.

13. Cgt. 14. Cgt. 15. Cgt. 16. Cgt. 17. Dgt. 18. Cgt.

19. Dgt. 20. (i) $0 < P < 1$ 20. (ii) $p > 2$

## EXERCISE 5.6

1. Divergent 2. Condinationally Cgt. 3. Dgt. 4. Conditionally Cgt.

5. Dgt. 6. Dgt. 7. Absolutely cgt. 8. Conditionally cgt. 9. Absolutely cgt.

10. Cond. cgt. 11. Dgt. 12. Dgt. 13. Cond. cgt. 14. Dgt. 15. Cond. cgt.

## EXERCISE 6.1

1. $x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$

2. $e^{-x} = \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{\cos x}{2} + \frac{\cos 2x}{5} + \frac{\cos 3x}{10} + \cdots\right)\right.$

$\left. + \left(\frac{\sin x}{2} + \frac{2 \sin 2x}{5} + \frac{3 \sin 3x}{10} + \cdots\right)\right]$

3. $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

4. $f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

5. $f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

6. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right]$

7. $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right] + \frac{\sin x}{2}$

8. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$

9. $f(x) = \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

10. $\frac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$

11. $f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

12. $x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

13. $x \sin x = -1 + \pi \sin x - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2 - 1}$

14. (i) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$

(ii) $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$

## EXERCISE 6.2

1. (i) $x = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}$ (ii) $f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

2. $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right]$

3. $x^2 = \frac{7}{3} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2} - \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}$

4. $x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right]$

$$+\frac{2}{\pi}\left[\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right]$$

5. $f(x) = -\frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\cos \pi x}{1^2} - \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} \cdots\right]$

$$+\frac{1}{\pi}\left[\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right]$$

6. $e^x = \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \cdots\right)\right.$

$$\left. -2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \cdots\right)\right]$$

7. $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots\right]$

8. $f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

9. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right)$

10. $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}[1-(-1)^n]\cos n\pi x$

11. $f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \cdots\right]$

$$-\frac{2}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right]$$

12. $f(x) = \frac{1}{4} + \frac{2l}{\pi}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \cdots\right]$

$$+\frac{l}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \cdots\right]$$

## EXERCISE 6.3

1. $x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$

2. $\cos x = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos 2nx}{1 - 4n^2}$

3. $x^3 = 2(-1)^n \left[\frac{6}{n^3} - \frac{\pi^2}{n}\right] \sum_{n=1}^{\infty} \sin nx$

4. $|x| = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos (2n-1)\pi x}{(2n-1)^2}$

5. $1 - x^2 = \frac{2}{3} + \frac{4}{\pi^2}\left[\cos \pi x - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right]$

6. $\cos px = \frac{2p}{\pi} \sin p\pi \left[\frac{1}{2p^2} + \frac{\cos x}{1 - p^2} - \frac{\cos 2x}{4 - p^2} + \frac{\cos 3x}{9 - p^2} - \cdots\right]$

7. $|\sin x| = \frac{2}{\pi}\left[1 - 2\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \cdots + \frac{\cos 2nx}{4n^2 - 1} + \cdots\right)\right]$

8. $f(x) = \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

9. $f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

## EXERCISE 6.4

2. $x = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

3. $2x - 1 = -\frac{8}{\pi^2}\left[\cos \pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \cdots\right]$

4. $e^x = 2\pi\left[\frac{1+e}{\pi^2+1}\sin\pi x + \frac{2(1-e)}{4\pi^2+1}\sin 2\pi x + \frac{3(1+e)}{9\pi^2+1}\sin 3\pi x + \cdots\right]$

5. $e^x = e-1+2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + \cdots\right]$

6. $f(x) = \frac{2}{\pi}\left[\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \cdots\right]$,

$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right]$.

7. $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} - \cdots\right]$,

$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 2x + \frac{\cos 6x}{9} + \frac{\cos 10x}{25} + \cdots\right]$

8. $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right]$

9. $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin\pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$

10. $x(\pi - x) = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]$

11. (i) $\pi - x = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

(ii) $\pi - x = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

(iii) $\pi - x = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

12. $\cos\left(\frac{\pi x}{p}\right) = \frac{1}{\pi}\left[\frac{4}{3}\sin\left(\frac{2\pi x}{p}\right) + \frac{8}{15}\sin\left(\frac{4\pi x}{p}\right) + \frac{12}{35}\sin\left(\frac{6\pi x}{p}\right) + \cdots\right]$

13. $\sin\left(\frac{\pi x}{\ell}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2\pi x/\ell)}{2^2-1} + \frac{\cos(4\pi x/\ell)}{4^2-1} + \cdots\right]$

## EXERCISE 7.1

1. 1, 1
2. 1, 1
3. 2, 1
4. 1, 3
5. 1, 6
6. 2, 4
7. 3, 5
8. 2, 1
9. 2, not defined
10. linear
11. non-linear
12. non-linear
13. linear
14. non-linear
15. non-linear

## EXERCISE 7.2

13. 2, –1
14. 1, 0, 2
15. $xy' - y + 3 = 0$
16. $(y')^2 - 4y = 0$
17. $(y')^3 = 4y(xy' - 2y)$
18. $\frac{d^2x}{dt^2} + n^2x = 0$
19. $y'' - 2y' + 2y = 0$
20. $x^2y'' - 3xy' + 3y = 0$
21. $y'' - y' - 2y + 2x + 1 = 0$.
22. $y'''[2 + 3(y')^2] = 9y'(y'')^2$
23. $x[yy'' + (y')^2] = yy'$
24. $2x\,yy' = y^2 - x^2$
25. $\frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}} \cdot y' = 0$
26. (i) $x\,dx + y\,dy = 0$, (ii) $xy'' - (y')^3 - y' = 0$
27. $(x^2 - y^2)\frac{dy}{dx} = 2xy$
28. $(x^2 + y^2)y'' = 2(xy' - y)(1 + y'^2)$
29. $[(1 + y'^2)]^3 = a^2(y')^2$
30. $xy' - 2y = 0$
31. (i) $4xy' = 4y + (y')^2$, (ii) $yy' = xy'^2 + 1$
32. $2a\,y'' + (y')^3 = 0$
33. $\frac{d^3y}{dx^3} = 0$

34. $xy\,y'' + x(y')^2 = yy'$

35. $x[yy'' + (y')^2] = yy'$

36. $y = -\frac{1}{4}\sin 2x + 1 + \frac{5}{2}x$

37. (i) $y = 3e^{4x} + 2e^{3x}$, (ii) $y = -2e^{4x}$

## EXERCISE 7.3

1. $(a + y)(a - x) = c$

2. $\tan^{-1}y = x - \log(1 + x) + c$

3. $e^x[\sin x + \cos x] = c$

4. $y = c \log x$

5. $\log y + y^2 = \sin x + 1$

6. $\sin^{-1}x - \sin^{-1}y = \frac{\pi}{3}$

7. $x^2 = y - \log(y + 1) + 2$

8. $e^y = \frac{1}{3}x^3 + e^x + c$

9. $1 - e^{-3y} = c\,e^{x^3}$

10. $\sin x \tan y + \sin(x + y) = c$

11. $y - \tan\left(\frac{x + y}{2}\right) = c$

12. $x^3 + y^3 = 3axy + c$

13. $\tan y = c(1 - e^x)^3$

14. $(y^2 + 1)(x^2 + 1) = c$

15. $\sin x - x\cos x = (y - 1)e^y + c$

16. $y = -[2\log(1 + x^2) + 4]^{1/2}$

17. $x = 2\cos y$

18. $y = \tan^{-1}(x + y + 1) + c$

19. $4x + y + 1 = 2\tan(2x + 2c)$

20. $c(y - a) = \frac{x}{1 + ax}$

21. $(x^2 + 1)\sin^2 y = 2$

22. $x\sin x + \cos x + 1 = y^6 - y$

## EXERCISE 7.4

1. $x = c\sin\left(\frac{y}{x}\right)$

2. $\log x + e^{-y/x} = c$

3. $x^2 - 2xy - y^2 = c$

4. $\tan^{-1}\left(\frac{y}{x}\right) = \log\left(\sqrt{x^2 + y^2}\right) + c$

5. $c(x^2 - y^2) = x$

6. $x(x^2 - 3y^2) = c$

7. $c(x + y)^2 = x\,e^{y/x}$

8. $x^2y = c^2(y + 2x)$

9. $\sqrt{xy} - x = c$

10. $x + y\,e^{x/y} = c$

11. $y^2 - x^2 = cx^3$

12. $(y - x)\,e^{y/x} = c$

13. $y^2(2x^2 + y^2) = c$

14. $y + x \log y = cx$

15. $y^4 + 6x^2y^2 + x^4 = c$

16. $y^2 + 2x^2 \log c\,y^2 = 0$

17. $x + (x + y) \log x = c(x + y)$

18. $y^3 + 6x^3 (\log x + c) = 0$

19. $\log x = \cos\left(\frac{y}{x}\right) + c$

20. $xy \cos\left(\frac{y}{x}\right) = c$

21. $y = x\,(\log x - 1)$

22. $y = -\sqrt{x^2 \log x^2 + 4x^2}$

23. $y + \sqrt{x^2 + y^2} = x^2$

24. $y = \sqrt{5x^3 - x^2}$

## EXERCISE 7.5

1. $x + y - 2 = c^2(x - y)^3$

2. $2x - y + \frac{1}{5}\log(5x - 15y + 17) = c$

3. $y - x + 3 = c(y + x + 1)^3$

4. $\log(2x + y - 1) + x + 2y = c$

5. $(x + y - 3)^5 = c(y - x - 1)$

6. $3x - 3y + c = 2 \log(3x + 6y - 1)$

7. $2x + 2y + 6 \log(2x + 3y - 7) = 2(4 + 3 \log 4)$

## EXERCISE 7.6

1. $x^2y^2 - x^3 = c$

2. $ax^2 + 2h\,xy + by^2 + 2gx + 2fy = c$

3. $3x^2y^3 + y^4 = c$

4. Not exact.

5. $x \sin y + y \cos x - \frac{1}{2}y^2 = c$

6. $y\,e^{-x} - \cos x + y^2 = c$

7. $xy - 2x\,e^x + 2\,e^x - 2x^3 = c$

8. $x^3y^3 - \tan^{-1}(3x) = c$

9. $x^3 + 2y \log 3x + 3 \log y = c$

10. $y(x + \log x) + x \cos y = c$

11. $x^3 - y \cos 2x + y + y^3 = c$

12. $x^2y^2 \cos x = c$

13. $x\,e^{2y} - \sin xy + y^2 + c = 0$

14. $e^{xy^2} + x^4 - y^3 = c$

15. $y \sin x + x^2 e^y - y = c$

16. $x^2y^2 - 2(x + y) + 2 = 0$

17. $x^2 - x \sin y = 4$

18. $x^3y^2 - y^3 x + x^2 + y + 1 = 0$

19. $y^2(1 - x^2) - \cos^2 x = 3$

20. $y^2 \cos x - y \sin^2 x - \frac{1}{2} y = \frac{15}{2} = 9$

21. $y^2 - 3y + 2x = 2xy$

22. $2x \tan y + \cos 2x = 2\pi + 1$

23. $2x^2 \cos y + 2x^3y - y^2 + 4 = 0$

## EXERCISE 7.7

1. $x + y^3 = cy$

2. $\frac{y}{x} + \sin \frac{1}{x} = c$

3. $ye^x + x^2 = cy$

4. $\frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{1}{4}x^4 + c = 0$

5. $y = x \tan (x + c)$

6. $y(c - x^2) = e^x$

7. $\frac{1}{xy} = \log \frac{x}{y} - c$

8. $\sqrt{x^2 + y^2} = y^3 + c$

9. $6e^x + 4x^3y - 3y^3 = 6cy$

10. $x^2 - y - 1 - x \cos y = cx$

11. $x^2 + y^2 = 2a^2 \tan^{-1}\left(\frac{y}{x}\right) + c$

12. $y + \log x + 1 + cx = 0$

13. $\frac{x}{y} + \log \frac{y^3}{x^2} = c$

14. $x^2 = 2cy + c^2$

## EXERCISE 7.8

1. $3y^3 \log y - x^3 = 3cy^3$

2. $2\log x - \log y - \frac{1}{xy} = c$

3. $4x^3 + 3x^4 + 6x^2y^2 = c$

4. $x^3y^3 + x^2 = cy$

5. $xy^2(x + 2y + 2) = c$

6. $x e^x + \sin y = c e^x$

7. $xy \log cx + 1 = 0$

8. $y \log x + x = cy$

9. $(3x^2y + y^3) e^{3x} = c$

10. $x^3y + \frac{1}{2}x^2y^2 = c$

11. $2x^2 e^{1/x^3} = 3y^2 + 6cx^2$

12. $y(x + y - 1) = c e^{-x}$

13. $xy^3 + 2x + y^4 = cy^2$

14. $x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$

15. $x^3y^2 + x^4y^3 = c$

16. $x^4y + x^3y^2 - \frac{1}{4}x^4 = c$

17. $\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$

18. $xy + y\cos y - \sin y = c$

19. $3x^4 - 8x^3 + 12x^2y^2 = c$

20. $\frac{1}{2}\left[xy + \log x - \frac{1}{xy}\right] - \frac{1}{2}\log y = c$

## EXERCISE 7.9

1. $y = \frac{1}{x^2}(c - x\cos x + \sin x)$

2. $y = \frac{1}{2}x e^{-2x} + \frac{c}{x}e^{-2x}$

3. $y \log x = (\log x)^2 + c$

4. $x = y^3 + cy$

5. $y = \tan x + c\, e^{\tan x} - 1$

6. $1 + x(y^2 - 2 + c\, e^{-y^2/2}) = 0$

7. $2r\sin^2\theta + \sin^4\theta = c$

8. $y = (x + 1)(e^x + c)$

9. $2xy(x^2 - 1) = 2x^2\log x + 2cx^2 + 1$

10. $y = \tan^{-1}x + c\, e^{-\tan^{-1}x} - 1$

11. $y = \sqrt{1 - x^2} + c(1 - x^2)$

12. $y^2 e^{2x} = c - x^2$

13. $\tan y = \frac{1}{2}(x^2 - 1) + c\, e^{-x^2}$

14. $y(x - 1)^4 = \frac{1}{3}x^3 - x + c$

15. $\sin y = (x + 1)(e^x + c)$

16. $6y = 2 - 3x + x^3$

17. $2y = e^{\pi x} + \sin x - \cos x$

18. $4x^3 = 3y(1 + x^2)$

19. $y\sec^2 x = \sec x - 2$

20. $2r = \sin 2\theta + \sqrt{2}\cos\theta$

21. $y^2 = x(1 + x) - 1$

## EXERCISE 7.10

1. $(y - 4x - c)(y - 3x - c) = 0$

2. $(y - x - c)(y^2 + x^2 - c) = 0$

3. $(y - e^x - c)(y - e^{-x} - c) = 0$

4. $(y + c)^2 = x(x - a)^2$

5. $(y - cx)(xy + x - c) = 0$

6. $(xy - c)(x^2 - y^2 - c) = 0$

7. $[y(1+\cos x)-c][y(1-\cos x)-c] = 0$

8. $(x^2 + y^2 - c)(y - ce^x) = 0$

9. $(y - cx^2)(y^2 + 3x^2 - c) = 0$

10. $(x^2 + 2y - c)(x + y + 1 - ce^x) = 0$

11. $(y - cx)(y^2 - x^2 - c) = 0$

12. $(y - ce^x)(2y - x^2 - c)(y + x - 1 - ce^{-x}) = 0$

## EXERCISE 7.11

1. $x = \log \frac{p}{1-p} - c$, $y = -\log(1-p) - c$

2. $y = (x - a)c - c^2$

3. $c^2x^3 - 2cxy + ax = 0$

4. $x = \frac{c}{p^2} - \frac{np^{n-1}}{n+1}$, $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$

5. $x = 2 - 2p + \frac{c}{e^p}$, $y = (1 + p)x + p^2$

6. $y = 2\sqrt{cx} + \tan^{-1}c$

7. $xy + c = c^2x$

8. $y = 2x\sqrt{c/x} + c^2$

9. $y = 2\sqrt{cx} + f(c)$

10. $x = -\frac{1}{p} + 2\tan^{-1}p + c$, $y = \log p + \log(1 + p^2)$

11. $x = \log\frac{p(p+1)}{p-1} + c$, $y = p - \log(p^2 - 1)$

12. $x(p-1)^2 = c - p + \log p$, $y(p-1)^2 = p^2(c - p + \log p) + p$.

## EXERCISE 7.12

1. $y = c - 2\left[\log(p - 1) + p + \frac{p^2}{2}\right]$, $x = c - 2[p + \log(p - 1)]$

2. $y^2 = 2cx + 4c^2$

3. $y = c - a\log(1 - p)$, $x = c + a\log\left(\frac{p}{1-p}\right)$

4. $y = \frac{3a}{2p^2} + \frac{2b}{p} + c$, $x = \frac{a}{p^3} + \frac{b}{p^2}$

5. $2y = p^2 - 2\log p - 2c$, $x = p + \frac{1}{p}$

6. $y^2 - (2x - m)c - nc^2 = 0$

7. $y^2 = 2cx + c^3$

8. $y\sqrt{p^2 - 1} = (p^2 - 1)^{3/2} + c$, $x = p(y + 3) - p^3$

9. $y = 4c(cxy + 1)$

10. $\log y = cx + c^2$

## EXERCISE 7.13

1. $y = c(x - a) - c^2$
2. $y = cx - e^c$
3. $y = cx + \frac{a}{c}$
4. $y = cx + \frac{n}{c} - mc$
5. $(y - cx)(c - 1) = c$
6. $y = cx - \sin^{-1} c$
7. $c(cx - y + k) = 3$
8. $y = cx \pm \sqrt{4 + 4c^2}$
9. $y^2 = cx^2 + c^2$
10. $y = 2c\sqrt{x} + f(c^2)$
11. $y^2 = cx^2 - \frac{a^2 c}{1 - c}$
12. $e^{2y} = ce^{2x} + c^2$
13. $y^2 = cx + \frac{1}{4} c^2$
14. $y^2 = cx^2 + \frac{a}{4c}$
15. $\sin y = c \sin x + c^2$.

## EXERCISE 7.14

1. $y^2 - (x - a) + 2cy + c^2 = 0$; $x - a = 0$
2. $(y + c)^2 = x(x - 1)^2$; $x = 0$
3. $y = cx + c^3$; $27y^2 + 4x^3 = 0$
4. $y = cx \pm \sqrt{1 + c^2}$; $x^2 + y^2 = 1$
5. $y = \frac{1}{2} cx^2 + \frac{1}{2c}$; $y = \pm x$
6. $y^2 = 2cx + c^3$; $27y^4 + 32x^3 = 0$
7. $(3y + 2c)^2 = 4cx^3$; $x^3 - 6y = 0$
8. $c^2(x^2 - 4) - 2cy - 1 = 0$; $x^2 + y^2 = 4$
9. $y^3 = 3cx + 6c^2$; $3x^2 + 8y^3 = 0$
10. $y = cx^2 + 2c^2$; $x^4 + 8y = 0$
11. $y^2 = cx^2 + c^2$; $4y^2 + x^4 = 0$
12. $c^2(x + y) - cxy - 1 = 0$; $x^2y^2 + 4(x + y) = 0$
13. $(x - c)y = c^2$; $y = 0$ and $y + 4x = 0$
14. $4x^2y + 1 = 0$
15. $y^2 + 4xy = 0$
16. $y = 1$
17. $y = 0, \pm 1$
18. $y = \pm \frac{\sqrt{2}}{3\sqrt{3}} x^2$

## EXERCISE 8.1

1. $y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$

2. $y = (c_1 + c_2 x) e^{2x} + c_3 e^{-2x}$

3. $y = (c_1 + c_2 x + c_3 x^2) e^{2x}$

4. $y = c_1 e^x + (c_2 + c_3 x) e^{-2x}$

5. $y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + c_4 \cos 2x + c_5 \sin 2x$

6. $y = c_1 e^x + c_2 \sin x + c_3 \cos x$

7. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

8. $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + (c_5 + c_6 x) e^{-x}$

9. $y = c_1 e^x + e^{2x} [c_2 \sin 3x + c_3 \cos 3x]$

10. $y = (c_1 + c_2 x) e^x + c_3 \cos x + c_4 \sin x$

11. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

12. $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

13. $y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x} + c_3 \cos x + c_4 \sin x$

14. $y = e^x [(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x]$

15. $y = 2e^x + e^{-2x}$

16. $y = 2e^{3x} - 5xe^{3x}$

17. $y = e^{5x}\left[\cos \sqrt{3}\, x - \frac{5\sqrt{3}}{3} \sin \sqrt{3}\, x\right]$

18. $y = e^{-2x} [2\sin x + \cos x]$

19. $y = e^x - 2e^{2x} + e^{3x}$

20. $y = e^{-3x} \sin 2x$

## EXERCISE 8.2

1. $y = c_1 e^{2x} + c_2 e^{3x} + xe^{3x}$

2. $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log (\sec ax + \tan ax)$

3. $y = c_1 + c_2 e^x + c_3 e^{-x} + \frac{1}{2} x e^x$

4. $y = (c_1 + c_2 x) e^{-x} + \frac{2}{9} e^{2x}$

## EXERCISE 8.3

1. $y = e^{2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$

2. $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \sinh x - \sin x$

3. $y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2} x + c_3 \sin\frac{\sqrt{3}}{2} x\right] + x e^{-x} + \frac{9}{2} - \frac{1}{4}(\cos x - \sin x)$

4. $y = c_1 e^x + (c_2 + c_3 x) e^{2x} + \frac{1}{2} x^2 e^{2x} - \frac{1}{18} e^{-x}$

5. $y = c_1 e^x + (c_2 + c_3 x) e^{-x} + \frac{1}{25}[2\cos 2x - \sin 2x]$

6. $y = c_1 e^x + e^x [c_2 \cos x + c_3 \sin x] + x e^x + \frac{1}{10}[3\sin x + \cos x]$

7. $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}\left[x^2 + \frac{5}{3} x + \frac{19}{18}\right] + \frac{1}{2} e^x \left(x + \frac{3}{2}\right)$

8. $y = c_1 e^{2x} + (c_2 + c_3 x) e^x + \frac{1}{9} x e^{-2x} + \frac{1}{6} x^2 e^x + \frac{1}{4} e^{-x}$

9. $y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x$

10. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} x \sinh x - \frac{2}{9} \cosh x$

11. $y = c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{64}(4x^2 - 9x) e^{2x}$

12. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{25}{108} x - \frac{1}{18} x^3 + \frac{1}{36} x^2$

13. $y = c_1 + (c_2 + c_3 x) e^{-x} + \frac{1}{18} e^{2x} + \frac{1}{3} x^3 - \frac{3}{2} x^2 + 4x$

14. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - \frac{1}{8} x^2 \cos x$

15. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x + \frac{1}{8}[1 - x \sin 2x]$

16. $y = (c_1 + c_2x)\, e^{2x} + 2x^2 + 4x + 3 + 4x^2\, e^{2x} + \cos 2x$

17. $y = c_1 \cos x + c_2 \sin x + e^{-x} + x^3 - 6x - e^x (2\cos x - \sin x) + 6 + \frac{1}{2} x \sin x$

18. $y = c_1 e^x + (c_2 + c_3x)\, e^{-2x} - \frac{1}{18}[x^3 + x^2]\, e^{2x}$

19. $y = e^{3x}[c_1 \cos 2x + c_2 \sin 2x] - \frac{2}{3} e^{3x} \sin 4x + \frac{2^x}{(\log 2)^2 - 6 \log 2 + 13}$

20. $y = (c_1 + c_2x)\, e^x - e^x[x \sin x + 2 \cos x]$

21. $y = 3 - \cos 2x - 2\cos x - 4\sin x$

22. $y = 6\cos x - \sin x + 3x^2 - 6 + 2x \cos x$

23. $y = \frac{3}{4} e^{-2x} - \frac{4}{5} e^{-3x} + \frac{1}{20} e^{2x}$

24. $y = 2e^{3x} - 6e^x + 3x^2 + 8x + 10$

25. $y = \cos x + 2\sin x + \frac{1}{4} x [\sin x - x \cos x]$

26. $y = \frac{1}{3} (\cos 3x - 2 \sin 3x) + \frac{29}{12} e^{3x} - \frac{1}{12} e^{-x} - \frac{2}{3} e^{2x}$

## EXERCISE 8.4

1. $y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{x}{2}$

2. $y = c_1x + c_2x^2 + \frac{1}{12x^2}$

3. $y = c_1x^4 + \frac{c_2}{x^5} - \frac{1}{14} x^2 - \frac{1}{9} x - \frac{1}{20}$

4. $y = c_1x^4 + \frac{c_2}{x} + \frac{1}{5} x^4 \log x - \frac{1}{2} \log x + \frac{3}{8}$

5. $y = [c_1 + c_2 \log (x + 1)]\, (x + 1)^2 + \frac{1}{2} (x + 1)^2 [\log (x + 1)]^2 - 2x - \frac{7}{4}$

6. $y = x[c_1 \cos (\log x) + c_2 \sin (\log x)] + x \log x$

7. $y = \frac{1}{x} [c_1 + c_2 \log x - \sin (\log x)]$

8. $y = c_1x^2 + (c_2 + c_3 \log x)\, x + \frac{1}{4} x^3 - \frac{3}{2} x (\log x)^2$

**9.** $y = c_1x^3 + \frac{c_2}{x} - \frac{x^2}{9}(3\log x + 2)$

**10.** $y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{1}{27} x^3 [\log x - 1]$

**11.** $y = x\left[c_1 \cos\left(\sqrt{3} \log x\right) + c_2 \sin\left(\sqrt{3} \log x\right)\right]$

$$+ \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{1}{2} x \sin(\log x)$$

**12.** $y = c_1 \cos[\log(x+1)] + c_2 \sin[\log(x+1)] - \frac{1}{3}\sin 2\{\log(x+1)\}$

**13.** $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\{(3x+2)^2 \log(3x+2) + 1\}$

**14.** $y = 2x^{-3} - x^2 + 2x^2 \log x$ **15.** $y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{2}\left[\log x - \frac{3}{2}\right]$

## EXERCISE 8.5

**1.** $y = c_1 \sin x + c_2 \cos x - x \cos x + \sin x \log \sin x$

**2.** $y = c_1(1 - x\cot x) + c_2 \cot x$ **3.** $y = c_1x + \frac{c_2}{x} + x^3$

**4.** $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log \tan\left(\frac{\pi}{4} + x\right)$

**5.** $y = \frac{1}{2} c_1 x e^{2x} + c_2x - \frac{1}{2}x^2$

**6.** $y = c_1e^x + c_2(2x+1)e^{-x} - \frac{1}{4}(2x^2 + 2x + 1)e^{-x}$

**7.** $y = e^{2x} + (c_1x^3 + c_2)e^x$

**8.** $y = c_1\left[\cos x - \frac{1}{2}\sec x\right] + c_2 \sin x + \frac{1}{2}\sec x$

**9.** $y = e^{-x}[c_1 + c_2x + \log(1 - e^{-x})]$ **10.** $y = c_1\left(x + \frac{1}{x}\right) + \frac{c_2}{x}$

**11.** $y = c_1x^3 + c_2x + x^3 \log x + x^2$ **12.** $y = e^{2x}[c_1(x-2)^2 + c_2]$

13. $y = -e^x - \frac{1}{4} c_1 (2x + 5) + c_2 e^{2x}$ **14.** $y = c_1 e^x + c_2 x - x^2 - 1$

15. $y = e^{2x} [x + c_1 \log(x + 1) + c_2]$

## EXERCISE 8.6

1. $y = (c_1 - x) \cos x + (c_2 + \log \sin x) \sin x$

2. $y = c_2 \cos 3x + c_1 \sin 3x + \frac{1}{9} \cos 3x \log \cos 3x + \frac{1}{3} x \sin 3x$

3. $y = c_2 \cos x + c_1 \sin x + \sin x \int \frac{\sin^3 x}{\cos^2 x} dx - \cos x \int \frac{\sin^4 x}{\cos^3 x} dx$

4. $y = e^x [c_1 - e^{-x} + \log(1 + e^x)] + e^{-x} [c_2 - \log(1 + e^x)]$

5. $y = c_1 \cos x + c_2 \sin x + \sin x \log(\text{cosec } x - \cot x) - \cos x \log(\sec x + \tan x)$

6. $y = c_1 \cos x + c_2 \sin x + \sin x \log(1 + \sin x) - x \cos x - \frac{\cos^2 x}{1 + \sin x}$

7. $y = e^{-2x} (c_1 \cos x + c_2 \sin x) + x e^{-2x} \sin x + e^{-2x} \cos x \log \cos x$

8. $y = c_1 x + c_2 x^{-1} + e^x \left(1 - \frac{1}{x}\right)$

9. $y = e^x (c_1 + c_2 x) + e^x \left[\sin^{-1} x (1 + x^2) + x \sqrt{1 - x^2} - \frac{x}{\sqrt{1 - x^2}}\right]$

10. $y = e^{3x} (c_1 + c_2 x) - e^{3x} \log x$

11. $y = c_2 e^{-x} + c_1 e^{-2x} - e^{-x} \log x - e^{-2x} \int \frac{e^x}{x} dx$

12. $y = c_1 e^x + c_2 x e^x + e^x \left[\frac{x^3 \log x}{6} - \frac{5x^3}{36}\right]$

13. $y = c_2 x + c_1 e^x - (x^2 + x + 1)$ **14.** $y = c_1 x + c_2 x e^{2x} - \frac{x^2}{2}$

## EXERCISE 8.7

1. $y = \frac{1}{2} c_1 \left[\sin^2 x \log \tan \frac{x}{2} - \cos x\right] + c_2 \sin^2 x$

2. $(x^2 + 1)\, y = c_1 x + c_2 - 2\cos x - \frac{x^3}{3}$

3. $xy\sqrt{x^2 - 1} = c_1\sqrt{x^2 - 1} + c_2 \log\left(x + \sqrt{x^2 - 1}\right) + \sec^{-1} x + c_3$

4. $(x^3 - 4x)\, y = x^3 + \frac{1}{2} c_1 x^2 + c_2 x + c_3$

5. $(x + \sin x)\, y = \frac{1}{2} c_1 x^2 + c_2 x + c_3 - \cos x$

6. $y = c_1 x + c_2 x\, e^{2/x} - \frac{1}{4}(x + 2)$

7. $y \tan x = c_1 (\tan x - x) + c_2$

## EXERCISE 8.8

1. $36y = 6x^3 \log x - 11x^3 + c_1 x^2 + c_2 x + c_3$

2.(i) $a^2 + c_1 y^2 = c_1^2 (x + c_2)^2$, (ii) $y = c_1 \sin(mx + c_2)$ 3. $y = \tan x$

4. $y = x^3 + c_1 x^2 + c_2$

5. $e^y \cos(x + c_1) = c_2$

6. $\pm c_1 x = c_2 + 2\sqrt{c_1 y - 1}$

7. $c_1 y = \tan(c_1 x + c_2)$ and $y = c$

8. $y = c_1 x^4 + c_2 x + c_3$

9. $y = x + c_1 \log(c_1 + y) + c_2$ and $y = c$

10. $y = \frac{1}{2}\left[\frac{c_1 x^2}{2} - \frac{1}{c_1} \log x\right] + c_2$

11. $y = (\sin^{-1} x)^2 + c_1 \sin^{-1} x + c_2$

12. $y = c_1 \sinh^{-1} x - 4x + c_2$

13. $y = \frac{1}{3} c_1 x^3 + 2c_1^2 x^2 + c_2 x + c_3$

14. $y = c_1$ and $y^2 + c_2 = c_3 e^{2x}$

15. $\log y = \sqrt{c_1} \sinh(x + c_2)$

16. $y = c_2 \pm \sqrt{c_1 - x^2}$

## EXERCISE 9.1

**1.** $x = -2c_1e^{2t} - \frac{1}{3}c_2 e^{3t}$

$y = c_1 e^{2t} + c_2 e^{3t}$

**2.** $y = e^{6t}[c_1 \cos t + c_2 \sin t]$

$x = \frac{1}{2}e^{6t}[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t$

**3.** $x = -t - \frac{2}{3}$

$y = \frac{1}{2}t^2 + \frac{4}{3}t + c$

**4.** $x = e^t + c$

$y = e^{-t} - e^t + \sin t$

**5.** $x = c_1 e^{-2t} + c_2 e^{-7t} - \frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196}$

$y = \frac{1}{3}\left[3c_2 e^{-7t} - 2c_1 e^{-2t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right]$

**6.** $y = c_1 + c_2 e^{2x} + e^x$

$z = c_1 - c_2 e^{-2x} + e^x$

**7.** $y = c_1 e^t + c_2 e^{-t}$; $z = c_2 e^{-t} - c_1 e^t$;

$x = 3c_1e^t - 3c_2e^{-t} + c_3$

**8.** $x = c_1e^{-5t} + c_2e^{-t/3} + \frac{1}{65}(8\sin t + \cos t)$

$y = -\frac{4}{3}c_1e^{-5t} + c_2e^{-t/3} + \frac{1}{130}[61\sin t - 33\cos t]$

**9.** $x = c_1e^t + c_2e^{-5t} + \frac{3}{7}e^{2t} - \frac{1}{25}(10t + 13)$

$y = c_1e^t - c_2e^{-5t} + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$

**10.** $x = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t$

$y = (\sqrt{2} + 1)c_1e^{\sqrt{2}t} + (1 - \sqrt{2})c_2e^{-\sqrt{2}t} + 2\sin t$

**11.** $x = c_1\cos 3t + c_2\sin 3t + \frac{3}{5}e^t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27}$

$y = \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{1}{2}(c_1 + c_2)\sin 3t - \frac{e^t}{5} - \frac{2t}{9}$

12. $x = c_1 + c_2 \sin t + c_3 \cos t + \frac{7}{10} e^{2t}$

$y = c_1 - c_2 \cos t + c_3 \sin t + \frac{1}{10} e^{2t}$

13. $x$ $+ c_2 t) e^t + (c_3 + c_4 t) e^{-t}$

$c_1$ $c_2 t) e^t - \frac{1}{2} (c_4$ $+ c_4 t) e^{-t}$

14. $x$ $c_1 + c_2 t) e^t + c_3 e^{2t} + 6t - 13$

$y = -3c_1 e^t - c_2 (1 + 3t) e^t - 5c_3 e^{2t} - 14t + 33$

15. $x = \frac{1}{4}(3e^t + 7e^{-t}) - \frac{1}{10}(19 \cos t - 2 \sin t) + \frac{2}{5} e^{2t}$

$y = -\frac{1}{12}(3e^t + 7e^{-t}) + \frac{1}{10}(19 \cos t - 2 \sin t) - \frac{1}{15} e^{2t}$

## EXERCISE 9.2

1. $xy + z = cy$
2. $x^2 + 2yz + 2z^2 = c$
3. $\log xyz + x + y + z = c$
4. $y + z = czx$
5. $(x + z)^2 = c(y + z)$
6. $2xy = y^2z^2 + cz^2$
7. $x^2y = cze^z$
8. $(\log x + \sin y) z = ce^z$
9. $x \log z = cy$
10. $xyz + x^2 + y^2 + z^2 = c$
11. $(x + y) z - y^2 = cz^2$
12. $x^2 + y^2 + \log z + \frac{z}{x} = c$
13. $x^3 + y^3 = e^{2z} + ce^z$
14. $x + y + z^2 = ce^{-x^2}$
15. $x^2 + y^2 + z(x + y) = cxy$
16. $x^2 - xy + y^2 = cz$

## EXERCISE 9.3

1. $z + \log x = c_1, x^2 + 2xy = c_2$
2. $x^2 + y^2 + z^2 = c_1, xyz = c_2$
3. $x = c_1 y, x = \log(yz - 2x) - \log y + c_2$
4. $xz = c_1, y - x + z = c_2$
5. $x^2 - y^2 = c_1, (x^2 - y^2) y^2 z = c_2 z - 2$

6. $x^2 - y^2 - z^2 = c_1, 2xy - z^2 = c_2$ 7. $x + y - z = c_1, (z - x)(z - y) = c_2$

8. $lx + my + nz = c_1, x^2 + y^2 + z^2 = c_2$ 9. $yz = c_1x, x^2 + y^2 + z^2 = c_2$

10. $x^2 + y^2 + z^2 = c_1, 2yz - y^2 + z^2 = c_2$

11. $y - 3x = c_1, 5x - \log[5z + \tan(y - 3x)] = c_2$

12. $(1 + x)^2 - (1 + y)^2 = c_1, 2 + x + y = c_2z$

13. $xy = c_1, x^4 - z^4 - 2xyz^2 = c_2$ 14. $x^3y^3z = c_1, \frac{y}{x^2} + \frac{x}{y^2} = c_2$

15. $xyz = c_1, x^2 + y^2 - 2z = c_2$ 16. $x + y = c_1z, 2y = c_2(y^2 - x^2)$

## EXERCISE 9.4 (a)

1. $z = px + qy + pq$ 2. $z = pq$

3. $yq + xp = 0$ 4. $px + qy = 0$

5. $2z = xp + yq$ 6. $yp - xq = 0$

7. $(y - z)p + (z - x)q = x - y$ 8. $\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}$

9. $xp - qy = x - y$ 10. $(p - q)z = y - x$

## EXERCISE 9.4 (b)

1. $f\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0$ 2. $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

3. $\phi\left(x^2 - z^2, \frac{y}{x + z}\right) = 0$ 4. $\phi(x^2 - y^2, x^2 - z^2) = 0$

5. $yz - y^2 = f(x^2 + y^2)$ 6. $f\left(x + y + z, \frac{x^2}{2} + yz\right) = 0$

7. $f(x + y + z, x^2 + y^2 - z^2) = 0$ 8. $x^2 + y^2 + z^2 = f(lx + my + nz)$

9. $x + y + z = f(xyz)$ 10. $f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$

**11.** $x^2 + y^2 + z^2 = z\,\phi\left(\frac{y}{z}\right)$

**12.** $\phi[x + y, \log(x^2 + y^2 + z^2 + 2xy) - 2x] = 0$

**13.** $\phi(xy, x^2 + y^2 + z^2) = 0$

**14.** $\phi(xy - z^2, x^2 - y - z) = 0$

**15.** $x^2 + y^2 + z^2 = f(xyz)$

**16.** $\phi\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$

**17.** $\phi(x^2 + y^2 - z^2, xy + z) = 0$

# References


1. Elementary Linear Algebra by J.A. Thorpe and P.G. Kumpel, Saunders College Publishing, New York.
2. Fundamentals of Linear Algebra by A.H. Lightstone, Appleton - Century - Crofts, Meredith Corporation, New York.
3. Linear Algebra by D.M. Schneider, Macmillan Publishing Company, New York.
4. Linear Algebra and Geometry by D.M. Bloom, Combridge University Press, London.
5. The Theory of Groups of finite order by W. Burnside, Dover Publications Inc. New York.
6. The Theory of Groups by M. Hall. Jr., The Macmillon Company, New York.
7. The Theory of Groups by A.G. Kurosch, Chelsea, New York.
8. The Theory of Groups by I.D. Mac Donald, Oxford University Press, New York.
9. The Theory of Groups by J.J. Rotman, Allyn & Bacon, Boston.
10. Topics in Algebra by I.N. Herstein, Xerox College Publishing, Tronto.
11. Theory of Groups by A. Majeed, Ilmi Kitab Khana, Lahore.
12. An Elementory Treatise on Differential Equations by H.T.H. Piaggio, G. Bell and Sons Ltd., London.
13. Advanced Mathematics for Engineers by H.W. Reddick, F.H. Miller, John Wesley and Sons, Inc, New York.
14. Applied Differential Equations by M.R. Spiegel, Prentice - Hall, Inc.
15. Differential Equations with Boundary Value Problems by Dennis G. Zill, PWS - KENT Publishing Co., Boston.
16. Elementary Differential Equations by E.I. Rainville, P.E. Bediant, Macmill Publishing Cor, Inc. New York.
17. Elements of Differential Equations by S.A. Rashid, Kitabistan Publishing Co., Lahore.
18. Introduction to Ordinary Differention Equations by S.L. Ross, John Wiley and Sons, New York.
19. Introductory Course in Differential Equations by D.A. Murray, Longman's Green and Ltd., London.
20. Mathematical Methods by M.C. Potter, J. Goldberg, P.H.I, Inc.
21. Mathematical Methods by S.M. Yusuf, A. Majeed, M. Amin, Ilmi Kitab Khana, Lahore.
22. Elements of Complex Variables by Louis L. Pennisi, Rinchant and Winston, N. York.
23. Introduction to Complex Analysis by R. Venonlinna, Addison Wesley Publishing Co., London.
24. Functions of a Complex Variable by Thomas M. Macrobert, Macmillon and Co. Ltd., London.
25. Complex Variables by H.R. Chilling Worth, Pergmanon Press, N. York.

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